MAT 201A RECAP

Jingyang Shu

1. A non-empty set $X$ with its metric $d$ is a metric space if $d : X \times X \to \mathbb{R}$ for $\forall x, y, z \in X$:
   (a) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
   (b) $d(x, y) = d(y, x)$;
   (c) $d(x, y) \leq d(x, z) + d(y, z)$.

2. A function $\| \cdot \| : X \to \mathbb{R}$ is called a norm if $\forall x, y \in X$ and $\lambda \in \mathbb{R}$:
   (a) $\| x \| \geq 0$ and $\| x \| = 0$ if and only if $x = 0$;
   (b) $\| \lambda x \| = |\lambda| \| x \|$;
   (c) $\| x + y \| \leq \| x \| + \| y \|$.

3. A function $f : X \to Y$ is sequentially continuous at $x \in X$ if $\forall \{x_n\} \subseteq X$ with $x_n \to x$ have $f(x_n) \to f(x)$ in $Y$.

4. Function $f$ is continuous at $x$ if and only if $f$ is sequentially continuous at $x$.

5. Function $f$ is continuous on $X$ if and only if $f^{-1}(G)$ is open for any open set $G \subseteq Y$.

6. A function $f : X \to \mathbb{R}$ is upper semicontinuous on $X$ if for all $x \in X$ and every sequence $x_n \to x$, we have $\limsup_{n \to \infty} f(x_n) \leq f(x)$; is lower semicontinuous if $\liminf_{n \to \infty} f(x_n) \geq f(x)$.

7. $(\tilde{X}, \tilde{d})$ is a completion of $(X, d)$ if:
   (a) $\exists i : X \to \tilde{X}$ such that $d(x, y) = \tilde{d}(i(x), i(y))$;
   (b) $i(X)$ is dense in $\tilde{X}$;
   (c) $(\tilde{X}, \tilde{d})$ is complete.

8. A subset $K$ of a metric space $X$ is sequentially compact if any sequence in $K$ has a convergent subsequence whose limit is in $K$.

9. $K$ is compact if and only if $K$ is sequentially compact if and only if $K$ is complete and totally bounded.

10. A sequentially compact metric space is separable.

11. A subset $A$ of a metric space $X$ is precompact if its closure in $X$ is compact, i.e., any sequence in $A$ has a Cauchy subsequence. Then, a set is compact if and only if it is closed and precompact.

12. Continuous functions:
   (a) send convergent sequences to convergent sequences (sequentially continuous);
   (b) send compact sets to compact sets;
   (c) on compact sets are uniformly continuous;
(d) attain maxima and minima in a compact set (when the value of function is in $\mathbb{R}$).

13. Closed subset of a complete space is also complete; closed subset of a compact space is also compact.

14. $C_c(X) \subset C_0(X) \subset C_b(X) \subset C(X)$, where

- $C_c(X) = \{ f \in C(X) | \text{supp } f \text{ is compact} \}$ where $\text{supp } f = \{ x \in X | f(x) \neq 0 \}$;
- $C_0(X)$ is the closure of $C_c(X)$;
- $C_b(X)$ is space of bounded continuous functions on $X$.

If $X$ is compact, then these spaces are equal.

15. (Weierstrass Approximation) The set of polynomials is dense in $C([a, b])$.

16. Let $\mathcal{F}$ be a family of functions from a metric space $(X, d)$ to a metric space $(Y, d)$. The family $\mathcal{F}$ is equicontinuous if for every $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon$ for all $f \in \mathcal{F}$. If $(X, d)$ is compact, then $\mathcal{F}$ is uniformly equicontinuous.

17. (Arzelà-Ascoli) If $K$ is a compact metric space, then a subset of $C(K)$ is compact if and only if it’s closed, bounded and equicontinuous.

18. A function $f : X \to \mathbb{R}$ on a metric space $X$ is Lipschitz continuous on $X$ if there is a constant $C \geq 0$ such that $|f(x) - f(y)| \leq Cd(x, y)$ for all $x, y \in X$. Every Lipschitz continuous function is uniformly continuous.

19. (Gronwall’s Inequality) Suppose that $u(t) \geq 0$ and $\varphi(t) \geq 0$ are continuous real-valued functions defined on the interval $0 \leq t \leq T$ and $u_0 \geq 0$ is a constant. If $u$ satisfies the inequality $u(t) \leq u_0 + \int_0^t \varphi(s)u(s)ds$ for $t \in [0, T]$, then $u(t) \leq u_0 \exp(\int_0^t \varphi(s)ds)$ for $t \in [0, T]$. In particular, if $u_0 = 0$ then $u(t) = 0$.

20. Suppose $f$ is continuous on $\mathbb{R}^2$, then for all $(t_0, u_0) \in \mathbb{R}^2$, there exists an open interval $I \subset \mathbb{R}$ with $t_0 \in I$, and a continuously differentiable function $u : I \to \mathbb{R}$ that satisfies the initial value problem:

\[
\begin{align*}
\frac{du}{dt} &= f(t, u) \\
u(t_0) &= u_0.
\end{align*}
\]

21. A collection $\mathcal{T}$ of subsets of $X$ is a topology if:

- (a) $\emptyset, X \in \mathcal{T}$;
- (b) if $G_\alpha \in \mathcal{T}$ for $\alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} G_\alpha \in \mathcal{T}$;
- (c) if $G_i \in \mathcal{T}$ for $i = 1, 2, \ldots, n$, then $\bigcap_{i=1}^n G_i \in \mathcal{T}$.

22. A subset $\mathcal{B}$ of a topology $\mathcal{T}$ is a base for $\mathcal{T}$ if for every $G \in \mathcal{T}$ there is a collection of sets $B_\alpha \in \mathcal{B}$ such that $G = \bigcup_\alpha B_\alpha$. A set $V \subset X$ is a neighborhood of a point $x \in X$ if there exists an open set $G \subset V$ with $x \in G$. A collection $\mathcal{N}$ of neighborhoods of a point $x \in X$ is called a neighborhood base for $x$ if for each neighborhood $V$ of $x$ there is a neighborhood $W \in \mathcal{N}$ such that $W \subset V$. A topology $\mathcal{T}$ on $X$ is called Hausdorff if every pair of distinct points $x, y \in X$ has a pair of nonintersecting neighborhoods.

23. A topological space $X$ is first countable if every $x \in X$ has a countable neighborhood base, and second countable if $X$ has a countable base.
24. A collection of open sets \( B \subset T \) is a base for the topology \( T \) on a set \( X \) if and only if \( B \) contains a neighborhood base of \( x \) for every \( x \in X \).

25. The closure \( A \) of a subset \( A \) of a topological space \( X \) is the smallest closed set that contains \( A \). The sequential closure \( \bar{A}^S \) is the set of limit points of sequences in \( A \). Note: \( A \subseteq \bar{A}^S \subseteq \bar{A} \).

If \( X \) is metrizable, then \( \bar{A}^S = \bar{A} \).

26. If \( T_1 \) and \( T_2 \) are two topologies on the same space \( X \), then the identity map \( I : (X, T_1) \to (X, T_2) \) is continuous if and only if \( T_1 \) is finer than \( T_2 \). Furthermore, \( I \) is homeomorphism if and only if \( T_1 = T_2 \).

27. Two metric topologies, defined by two metrics on the same space, are equal if and only if they have the same collection of convergent sequences with the same limits.

28. Let \( X \) and \( Y \) be two spaces, each with two topologies \( T_1, T_2 \) and \( S_1, S_2 \) respectively. Suppose that \( f : (X, T_1) \to (Y, S_1) \) is continuous. Then if \( T_2 \) finer than \( T_1 \), then \( f : (X, T_2) \to (Y, S_1) \) is continuous; if \( S_2 \) coarser than \( S_1 \), then \( f : (X, T_1) \to (Y, S_2) \) is continuous.

29. Compact sets of topological spaces are not necessarily closed. Compact sets of Hausdorff spaces are closed.

30. (Baire Category Theorem) If \((X, d)\) is a complete metric space and \( A_n \subset X \) for all \( n \geq 1 \) satisfying \( X = \bigcup_{n \geq 1} A_n \), then not all of the \( A_n \) are nowhere dense, i.e., there is at least one \( n \) such that \( \overline{A_n} \) has a non-empty interior.

31. For \( T \in \mathcal{B}(X, Y) \), \( \|T\| = \inf \{ M \geq 0 \|Tx\| \leq M\|x\|, \forall x \in X \} = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\| = 1} \|Tx\| = \sup_{\|x\| \leq 1} \frac{\|Tx\|}{\|x\|} \).

32. If \( X \) and \( Y \) are normed linear spaces, let \( T : X \to Y \) be linear, then \( T \) is bounded if and only if \( T \) is continuous.

33. (Bounded Linear Transformation, or B.L.T.) If \( X \) is a normed linear space and \( Y \) is a Banach space, if \( M \subset X \) is a dense linear subspace and \( T : M \to Y \) is a bounded linear map, then there exists unique \( \overline{T} : X \to Y \) is a bounded linear transformation with \( \overline{T}(x) = T(x) \) for \( x \in M \) and \( \|\overline{T}\| = \|T\| \).

34. Two norms on a linear space generate the same topology if and only if the norms are equivalent.

35. (Open Mapping Theorem) Let \( T \in \mathcal{B}(X, Y) \), for Banach spaces \( X \) and \( Y \). If \( T \) is one-to-one and onto, then \( T^{-1} \in \mathcal{B}(Y, X) \).

36. If \( X \) and \( Y \) are normed linear spaces, and \( T \in \mathcal{B}(X, Y) \), then \( \ker T \) and \( \text{ran} T \) are subspaces of \( X \) and \( Y \) respectively. Moreover, \( \ker T \) is closed.

37. Let \( X \) and \( Y \) be Banach spaces and \( T \in \mathcal{B}(X, Y) \), then \( \ker T = \{0\} \) and \( \text{ran} T \) is closed if and only if \( \exists c > 0 \) such that \( \forall x \in X : c\|x\| \leq \|Tx\| \).

38. Let \( X \) and \( Y \) be normed linear spaces and \( T \in \mathcal{B}(X, Y) \), \( T \) is called compact if \( T \) maps bounded subsets of \( X \) into precompact subsets in \( Y \). Equivalently, \( T \) is compact if for all bounded sequence \( \{x_n\} \) in \( X \), the sequence \( \{Tx_n\} \) has a convergent subsequence in \( Y \).
39. In finite-dimensional linear spaces:
   (a) the space itself is a Banach space;
   (b) every linear operator is bounded;
   (c) any two norms are equivalent.
40. A sequence \( \{T_n\} \) in \( B(X,Y) \) converges strongly if \( \lim_{n \to \infty} T_n x = Tx \) for every \( x \in X \).
41. Uniform convergence of operators implies strong convergence.
42. (Hahn-Banach) If \( Y \) is a linear subspace of a normed linear space \( (X, \|\cdot\|) \) and \( \psi : Y \to \mathbb{R} \) is a bounded functional on \( Y \) with \( \|\psi\| = M \), then there is a bounded linear functional \( \varphi : X \to \mathbb{R} \) on \( X \) such that \( \varphi|_Y = \psi \) and \( \|\varphi\| = M \).
43. For \( x, y \in X \), if \( \varphi(x) = \varphi(y) \) for all \( \varphi \in X^* \), then \( x = y \).
44. A sequence \( \{x_n\} \) in a Banach space \( X \) converges weakly to \( x \), denoted by \( x_n \rightharpoonup x \) as \( n \to \infty \), if \( \varphi(x_n) \to \varphi(x) \) as \( n \to \infty \) for every bounded linear functional \( \varphi \in X^* \).
45. The dual space \( X^{**} \) of \( X^* \) is called the bidual of \( X \). If \( X^{**} \cong X \), then \( X \) is reflexive. In particular, every Hilbert space is reflexive \( (\mathcal{H} \cong \mathcal{H}^* \cong \mathcal{H}^{**}) \).
46. Let \( X^* \) be the dual of a Banach space \( X \). We say \( \varphi \in X^* \) is the weak-\( * \) limit of a sequence \( \{\varphi_n\} \) in \( X^* \) if \( \varphi_n(x) \to \varphi(x) \) as \( n \to \infty \) for every \( x \in X \). We denote weak-\( * \) convergence by \( \varphi_n \rightharpoonup \ast \varphi \).
47. (Banach-Alaoglu) The closed unit ball in \( X^* \) is compact in the weak-\( * \) topology.
48. Let \( X \) be a linear space over \( \mathbb{C} \) (or \( \mathbb{R} \)), then an inner product on \( X \) is \( (\cdot, \cdot) : X \times X \to \mathbb{C} \) (or \( \mathbb{R} \)) such that:
   (a) linear in the second argument: \( (x, y + \lambda z) = (x, y) + \lambda (x, z) \) for all \( x, y, z \in X, \lambda \in \mathbb{C} \) (or \( \mathbb{R} \));
   (b) Hermitian symmetric: \( (x, y) = \overline{(y, x)} \), for all \( x, y \in X \);
   (by (a) and (b): \( (\lambda x + \mu y, z) = \overline{\lambda} (x, z) + \overline{\mu} (y, z) \) for all \( x, y, z \in X \) and \( \lambda, \mu \in \mathbb{C} \) (or \( \mathbb{R} \))
   (c) non-negative: \( (x, x) \geq 0 \), for all \( x \in X \);
   (d) non-degenerate: for all \( x \in X \): \( (x, x) = 0 \) if and only \( x = 0 \).
49. (Cauchy-Schwarz) If \( x, y \in X \), where \( X \) is an inner product space with its induced norm \( \|\cdot\| \), then \( |(x, y)| \leq \|x\| \|y\| \)
50. A normed linear space \( (X, \|\cdot\|) \) is an inner product space with the norm derived from the inner product if and only if (parallellogram law): \( \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \) for all \( x, y \in X \). Moreover, if the parallellogram law holds, then the inner product (polarization):
   \( (x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 - i\|x + iy\|^2 + i\|x - iy\|^2) \).
51. For any subset \( A \) of a Hilbert space \( \mathcal{H} \), the orthogonal complement \( A^\perp \) is always a closed linear subspace of \( \mathcal{H} \).
52. (Projection Theorem) Let \( \mathcal{M} \) be a closed linear subspace of a Hilbert space \( \mathcal{H} \), then
   (a) For each \( x \in \mathcal{H} \) there is a unique closest point \( y \in \mathcal{M} \) such that \( \|x - y\| = \min_{z \in \mathcal{M}} \|x - z\| \);
53. If $\mathcal{M}$ is a subspace of a Hilbert space $\mathcal{H}$, then $\mathcal{H} = \overline{\mathcal{M}} \oplus \mathcal{M}^\perp$.

54. Let $\{x_\alpha | \alpha \in I\}$ be an indexed set in a Banach space $X$, where the index set $I$ may be countable or uncountable. For each finite subset $J$ of $I$, we define the **partial sum** $S_J$ by $S_J = \sum_{\alpha \in J} x_\alpha$. The **unordered sum** of the indexed set $\{x_\alpha | \alpha \in I\}$ converges to $x \in X$, written $x = \sum_{\alpha \in I} x_\alpha$, if for every $\epsilon > 0$ there is a finite subset $J^\epsilon$ of $I$ such that $\|S_J - x\| < \epsilon$ for all finite subsets $J$ of $I$ that contain $J^\epsilon$, then the unordered sum is said to be **converge unconditionally**. An unordered sum is **Cauchy** if for every $\epsilon > 0$ there is a finite set $J^\epsilon \subset I$ such that $\|S_K\| < \epsilon$ for every finite set $K \subset I \setminus J^\epsilon$.

55. \[
\left( \sum_{\alpha \in I} x_\alpha, \sum_{\beta \in J} y_\beta \right) = \sum_{(\alpha, \beta) \in I \times J} (x_\alpha, y_\beta).
\]

56. An unordered sum in a Banach space converges if and only if it is Cauchy.

57. Let $U = \{u_\alpha | \alpha \in I\}$ be an orthogonal subset of a Hilbert space $\mathcal{H}$, then $\sum_{\alpha \in I} u_\alpha$ converges unconditionally if and only if $\sum_{\alpha \in I} \|u_\alpha\|^2 < \infty$. Moreover, $\left\| \sum_{\alpha \in I} u_\alpha \right\|^2 = \sum_{\alpha \in I} \|u_\alpha\|^2$.

58. (Bessel’s Inequality) Let $U = \{u_\alpha | \alpha \in I\}$ be an orthonormal set in a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$. Then:

(a) $\sum_{\alpha \in I} |\langle u_\alpha, x \rangle|^2 \leq \|x\|^2$;

(b) $x_U = \sum_{\alpha \in I} \langle u_\alpha, x \rangle u_\alpha$ is a convergent sum;

(c) $x - x_U \in U^\perp$.

59. For any subset $U \subset \mathcal{H}$, denote: $[U] = \{ \sum_{u \in U} c_u u | c_u \in \mathbb{C} \text{ and } \sum_{u \in U} c_u u \text{ converges unconditionally} \}$. Then $[U] = \text{span } U$.

60. If $U = \{u_\alpha | \alpha \in I\}$ is an orthonormal subset of a Hilbert space $\mathcal{H}$, then the following are equivalent:

(a) $\langle u_\alpha, x \rangle = 0$ for all $\alpha \in I$ implies $x = 0$;

(b) $x = \sum_{\alpha \in I} \langle u_\alpha, x \rangle u_\alpha$ for all $x \in \mathcal{H}$;

(c) (Pareval’s identity) $\|x\|^2 = \sum_{\alpha \in \mathcal{H}} |\langle u_\alpha, x \rangle|^2$ for all $x \in \mathcal{H}$;

(d) $[U] = \mathcal{H}$

(e) $U$ is a maximal orthonormal set.

61. Suppose that $U = \{u_\alpha | \alpha \in I\}$ is an orthonormal basis of a Hilbert space $\mathcal{H}$. If $x = \sum_{\alpha \in I} a_\alpha u_\alpha$ and $y = \sum_{\alpha \in I} b_\alpha u_\alpha$, where $a_\alpha = \langle u_\alpha, x \rangle$ and $b_\alpha = \langle u_\alpha, y \rangle$, then $\langle x, y \rangle = \sum_{\alpha \in I} \overline{a_\alpha} b_\alpha$. 

5
62. In a linear space $X$, a linear map $P$ is a projection if $P^2 = P$. Then $X = \text{ran } P \oplus \ker P$. Conversely, if $X = M \oplus N$, where $M$ and $N$ are linear subspaces of $X$, then there is a projection $P : X \to X$ with $\text{ran } P = M$ and $\ker P = N$.

63. If $\mathcal{H}$ is a Hilbert space, then an orthogonal projection $P : \mathcal{H} \to \mathcal{H}$ such that $P^2 = P$, and $\langle Px, y \rangle = \langle x, Py \rangle$ for all $x, y \in \mathcal{H}$. $1 - P$ is also an orthogonal projection. If $P$ is nonzero, $\|P\| = 1$.

64. (Riesz Representation Theorem) If $\varphi \in \mathcal{H}^*$, then there exists a unique $y \in \mathcal{H}$ such that $\varphi(x) = (y, x), \forall x \in \mathcal{H}$. Furthermore, $\|\varphi\| = \|y\|$. 
