MAT 201B RECAP

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1. Let $\Omega$ be a non-empty set. A collection $\Sigma$ of subsets of $\Omega$ is called $\sigma$-algebra if:
   (a) If $A \in \Sigma$, then $A^c \in \Sigma$;
   (b) If $A_1, A_2, \cdots$ is a countable family of sets in $\Sigma$, then their union $\bigcup_{i=1}^{\infty} A_i \in \Sigma$;
   (c) $\Omega \in \Sigma$.

2. Let $\Sigma$ be a $\sigma$-algebra on $\Omega$, a measure $\mu$ on $\Omega$ is a function $\mu : \Sigma \to [0, +\infty]$ satisfying:
   (a) $\mu(\emptyset) = 0$;
   (b) $\mu(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for $A_i \in \Sigma$ and $A_i \cap A_j = \emptyset$ if $i \neq j$.

3. A measure space is $\sigma$-finite if there exist countably many sets $A_i \in \Sigma$ with $\mu(A_i) < \infty$ and $\Omega = \bigcup_{i=1}^{\infty} A_i$.

4. Let $(\Omega, \Sigma, \mu)$ be a measure space. If $f : \Omega \to \mathbb{R}$, then we say that $f$ is a measurable function if for every $t \in \mathbb{R}$, the level set: $S_f(t) = \{x \in \Omega | f(x) > t\}$ is measurable. $f : \Omega \to \mathbb{C}$ is measurable if and only if both $\text{Re}(f)$ and $\text{Im}(f)$ are measurable.

5. A real function $f : \Omega \to \mathbb{R}$ is lower semicontinuous on $\Omega$ if $S_f(t)$ is open, and is upper semicontinuous if $\{x \in \Omega | f(x) < t\}$ is open. Equivalently, $f$ is lower semicontinuous if $\forall \epsilon > 0, \exists$ open ball $B \subset \Omega$ containing $x$ with $f(y) \geq f(x) - \epsilon$ for every $y \in B$ (equivalently, $f(x) \leq \liminf_{n \to \infty} f(x_n)$ for all $x_n \to x$), and is upper semicontinuous if $f(y) \leq f(x) + \epsilon$ (equivalently, $f(x) \geq \limsup_{n \to \infty} f(x_n)$ for all $x_n \to x$).

6. Consider measure space $(\mathbb{R}^n, B, \mu)$, and $f$ is a Borel measurable function. Let $\Omega = \{A \subset \mathbb{R}^n | A$ is open and $f(x) = 0$ for $\mu$-a.e. $x \in A\}$, then the essential support of $f$ is $(\bigcup_{A \in \Omega} A)^c$, which is a closed set.

7. Let $(\Omega, \Sigma, \mu)$ be a measure space, and $f : \Omega \to \mathbb{R}^+$ be measurable, then the integral of $f$ is $\int_{\Omega} f(x) d\mu(x) := \int_{0}^{\infty} \mu(%\{x \in \Omega | f(x) > t\}) dt$. If this quantity is less than $\infty$, we say $f$ is integrable.

8. (Monotone Convergence Theorem) Let $(\Omega, \Sigma, \mu)$ be a measure space and $\{f_j\}_{j=1}^{\infty}$ be an increasing sequence of integrable functions. Define $A_j := \{x \in \Omega | f_{j+1}(x) < f_j(x)\} \in \Sigma$ and $A := \bigcup_{j=1}^{\infty} A_j \in \Sigma$. If $f(x) := \begin{cases} \lim_{j \to \infty} f_j(x) & x \in A^c \\ 0 & x \in A \end{cases}$ and $I := \lim_{j \to \infty} \int_{\Omega} f_j(x) d\mu(x)$. Then
   (a) $f$ is measurable;
   (b) $I$ if finite if and only if $f$ is integrable, and in this case, $I = \int_{\Omega} f(x) d\mu(x)$, i.e., $\lim_{j \to \infty} f_j(x) d\mu(x) = \int_{\Omega} f(x) d\mu(x)$.

1 Last updated: March 22, 2015
9. (Fatou’s Lemma) Let \((\Omega, \Sigma, \mu)\) be a measure space, and \(\{f_j\}\) be a sequence of non-negative, integrable functions. Then \(\liminf_{j \to \infty} \int_{\Omega} f_j(x) d\mu(x) \geq \int_{\Omega} \liminf_{j \to \infty} f_j(x) d\mu(x)\).

10. (Dominated Convergence) Let \((\Omega, \Sigma, \mu)\) be a measure space. Let \(\{f_j\}\) be a sequence of (complex-valued,) integrable, and pointwise converging to \(f\) \(\mu\)-a.e., if there exists \(G\) on \((\Omega, \Sigma, \mu)\) which is integrable and \(|f_j(x)| \leq G(x), \forall j \geq 1\). Then \(\lim_{j \to \infty} \int_{\Omega} f_j(x) d\mu(x) = \int_{\Omega} f(x) d\mu(x)\).

11. (Tonelli-Fubini) Let \((\Omega_1, \Sigma_1, \mu_1)\) and \((\Omega_2, \Sigma_2, \mu_2)\) be two \(\sigma\)-finite measure spaces. Let \(f\) be \(\Sigma = \Sigma_1 \times \Sigma_2\) measurable on \(\Omega = \Omega_1 \times \Omega_2\). Then
   
   (a) (Tonelli) If \(f \geq 0\), then
   
   \[
   \int_{\Omega_1 \times \Omega_2} f(x, y) d(\mu_1 \times \mu_2)(x, y) = \int_{\Omega_1} \left( \int_{\Omega_2} f(x, y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_2} \left( \int_{\Omega_1} f(x, y) d\mu_1(x) \right) d\mu_2(y)
   \]

   Note: Above three values can be \(\infty\).

   (b) (Fubini) If \(f\) is complex-valued and \(\int_{\Omega_1 \times \Omega_2} |f(x, y)| d(\mu_1 \times \mu_2)(x, y) < \infty\), then all three integrals above are finite and equal.

12. A measurable function \(f : \Omega \to \mathbb{C}\) is said to be a \textbf{simple function} if it takes only finitely many values, i.e., \(\exists N \geq 1, c_1, c_2, \cdots, c_N \in \mathbb{C}\) and \(A_1, A_2, \cdots, A_N \in \Sigma\) with \(f(x) = \sum_{j=1}^{N} c_j \chi_{A_j}\)

13. Let \((\Omega, \Sigma)\) be a non-empty set and \(\sigma\)-algebra.

   (a) Let \(f : \Omega \to [0, \infty]\) be measurable, then \(\exists \{g_n\}\) of simple functions with (i) \(0 \leq g_1 \leq g_2 \leq \cdots \leq f\), (ii) \(g_n \to f\) pointwise and (iii) \(g_n \to f\) uniformly on \(A \in \Sigma\) if \(f\) is bounded on \(A\);

   (b) Let \(f : \Omega \to \mathbb{C}\) be measurable, then \(\{g_n\}\) of simple functions with (i) \(0 \leq |g_1| \leq |g_2| \leq \cdots \leq |f|\), (ii) \(g_n \to f\) pointwise and (iii) \(g_n \to f\) uniformly on any \(A \in \Sigma\) on which \(f\) is bounded.

14. Let \(f, g \in C(\mathbb{T})\), then the \textbf{convolution} of \(f\) and \(g\) is \(f \ast g : \mathbb{T} \to \mathbb{C}\), with \((f \ast g)(x) = \int_{\mathbb{T}} f(x-y)g(y)dy\).

15. \((f \ast g)_n = \sqrt{2\pi} f_n g_n\).

16. A family of functions \(\{\varphi_n\}_{n \geq 1} \subset C(\mathbb{T})\) is an \textbf{approximation identity} if

   (a) \(\varphi_n(x) \geq 0, \forall n, x;\)

   (b) \(\int_{\mathbb{T}} \varphi_n(x) dx = 1;\)

   (c) \(\forall \delta > 0, \lim_{n \to \infty} \int_{|x| \leq \delta} \varphi_n(x) dx = 0.\)

17. If \(f, g \in L^2(\mathbb{T})\), then \(f \ast g \in C(\mathbb{T})\) and \(\|f \ast g\|_\infty \leq \|f\|_2 \|g\|_2\).

18. Fourier coefficients: \(\hat{f}_n = \langle e_n, f \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-inx} f(x) dx\), where \(e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}\). Then \(f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}.\)
19. If \( f \in C^k(\mathbb{T}) \), then \( \hat{f}^{(k)}(n) = (in)^k \hat{f}(n) \).

20. The **Sobolev space** \( H^1(\mathbb{T}) \) consists of all functions \( f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} \in L^2(\mathbb{T}) \) such that
\[
\sum_{n=-\infty}^{\infty} n^2 |\hat{f}(n)|^2 < \infty.
\]
For such \( f \) we define the **weak derivative** of \( f \): \( f'(x) = \sum_{n=-\infty}^{\infty} in\hat{f}(n) e^{inx} \).

This Sobolev space is a Hilbert space with respect to \( \langle f, g \rangle_{H^1} = \langle f, g \rangle_{L^2} + \langle f', g' \rangle_{L^2} = \sum_{n \in \mathbb{Z}} (1 + n^2) \hat{f}(n) \bar{\hat{g}}(n) \).

21. The Sobolev space \( H^k(\mathbb{T}) \) is defined by: \( H^k(\mathbb{T}) := \{ f \in L^2(\mathbb{T}) | \sum_{n \in \mathbb{Z}} n^{2k} |\hat{f}(n)|^2 < \infty \} \). If \( k > \frac{1}{2} \), set \( S_N(x) := \sum_{n=-N}^{N} \hat{f}(n) e_n(x) \), then there exists \( c_k < \infty \) independent of \( f \) for which
\[
\|S_N - f\|_\infty \leq \frac{c_k}{N^{k-\frac{1}{2}}} \left( \sum_{n \in \mathbb{Z}} n^{2k} |\hat{f}(n)|^2 \right)^{\frac{1}{2}}.
\]

22. (Sobolev Embedding Theorem) If \( f \in H^k(\mathbb{T}) \) for \( k > \frac{1}{2} \), then \( f \in C(\mathbb{T}) \).

23. The **circle map** is a map \( F_\gamma : \mathbb{T} \to \mathbb{T} \) with \( F_\gamma(x) = x + 2\pi \gamma \). For each \( x_0 \in \mathbb{T} \), the iterated application of \( F_\gamma \) generates a sequence of points \( (x_n)_{n=0}^\infty \), where \( x_n = F_\gamma(x_{n-1}) \). The set \( \{x_n\} \) is called the **orbit** or **trajectory** of \( x_0 \) under \( F_\gamma \).

24. Let \( f : \mathbb{T} \to \mathbb{C} \) be a continuous function on \( \mathbb{T} \). The **time average** is \( \langle f \rangle_t(x_0) = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} f(x_n) \).

The **phase-space average** is \( \langle f \rangle_{\text{ph}} = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) dx \).

25. (Weyl’s Ergodic Theorem) If \( \gamma \) is irrational, then \( \langle f \rangle_t(x_0) = \langle f \rangle_{\text{ph}} \) for all \( f \in C(\mathbb{T}) \) and all \( x_0 \in \mathbb{T} \).

26. Suppose that \( \gamma \) is irrational and \( I \) is an interval in \( \mathbb{T} \) of length \( \lambda \). Then
\[
\lim_{N \to \infty} \frac{\#\{ n | 0 \leq n \leq N, x_n \in I \} }{N+1} = \frac{\lambda}{2\pi}.
\]

27. An **orthogonal projection** on a Hilbert space \( \mathcal{H} \) is a linear map \( P : \mathcal{H} \to \mathcal{H} \) that satisfies \( P^2 = P \) and \( \langle Px, y \rangle = \langle x, Py \rangle \) for all \( x, y \in \mathcal{H} \). If \( P \) is a nonzero orthogonal projection, then \( \|P\| = 1 \).

28. (Riesz Representation) If \( \varphi \) is a bounded linear functional on a Hilbert space \( \mathcal{H} \), then there is a unique vector \( y \in \mathcal{H} \) such that \( \varphi(x) = \langle y, x \rangle \) for all \( x \in \mathcal{H} \).

29. If \( A : \mathcal{H} \to \mathcal{H} \) is a bounded linear operator, then \( \text{ran } A = (\ker A^*)^\perp \) and \( \ker A = (\text{ran } A^*)^\perp \).

30. A bounded linear operator \( A : \mathcal{H} \to \mathcal{H} \) on a Hilbert space \( \mathcal{H} \) satisfies the **Fredholm alternative** if one of the following holds:
(a) \( Ax = 0 \), \( A^*x = 0 \) have only the zero solution, and the equations \( Ax = y \), \( A^*x = y \) have a unique solution \( x \in \mathcal{H} \) for every \( y \in \mathcal{H} \);
33. If \( A \) is Fredholm and \( K \) is compact, then \( A + K \) is Fredholm, and \( \text{ind} \ (A + K) = \text{ind} \ A \).

34. Suppose that \( A : \mathcal{H} \to \mathcal{H} \) is a bounded linear operator on a Hilbert space \( \mathcal{H} \) with closed range. Then the equation \( Ax = y \) has a solution for \( x \) if and only if \( y \) is orthogonal to \( \ker A^* \).

35. If \( A \) is a bounded self-adjoint operator on a Hilbert space \( \mathcal{H} \), then \( \| A \| = \sup \langle x, Ax \rangle \).

36. If \( A \) is a bounded self-adjoint operator on a Hilbert space \( \mathcal{H} \), then polarization identity:

\[
\langle y, Ax \rangle = \frac{1}{4} \left( \langle x + y, A(x + y) \rangle - \langle x - y, A(x - y) \rangle - i \langle x + iy, A(x + iy) \rangle - i \langle x - iy, A(x - iy) \rangle \right).
\]

37. A linear mapping \( U : \mathcal{H}_1 \to \mathcal{H}_2 \) between (real or) complex Hilbert spaces is said to be (orthogonal or) unitary if \( U \) is invertible and \( \langle Ux, Uy \rangle_{\mathcal{H}_2} = \langle x, y \rangle_{\mathcal{H}_1}, \forall x, y \in \mathcal{H}_1 \). For unitary operators, \( \|U\| = 1 \). An operator \( U : \mathcal{H} \to \mathcal{H} \) is unitary if and only if \( U^*U = I \).

38. Let \( T : \mathcal{H} \to \mathcal{H} \) is said to be \textbf{normal} if it commutes with its adjoint, i.e., \( TT^* = T^*T \).

39. (Mean Ergodic Theorem) Let \( \mathcal{H} \) be a Hilbert space and \( U \) unitary on \( \mathcal{H} \). Let \( \mathcal{M} = \{ x \in \mathcal{H} | Ux = x \} \) called invariant subspace for \( U \). Let \( P \) be the orthogonal projection onto \( \mathcal{M} \). Then

\[
\forall x \in \mathcal{H}, \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} U^n x = Px.
\]

40. A one-to-one, onto, and measurable (i.e., \( T^{-1}(A) \in \Sigma, \forall A \in \Sigma \)) mapping \( T : \Omega \to \Omega \) is said to be \textbf{measure preserving} if \( P(T^{-1}(A)) = P(A) \) for all \( A \in \Sigma \).

41. A mapping \( f : \Omega \to \mathbb{C} \) that is measurable is called a \textbf{random variable}. A measure preserving mapping \( T \) on a probability space \( (\Omega, \Sigma, P) \) is said to be \textbf{ergodic} if the only functions \( f \in L^2(\Omega, dP) \) satisfying \( f \circ T = f \) are the constants.

42. A one-to-one, onto, measure preserving map \( T : \Omega \to \Omega \) on a probability space \( (\Omega, P) \) is ergodic if and only if for every \( f \in L^2(\Omega, P) \),

\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} f \circ T^n = \int_{\Omega} f dP,
\]

where the convergence is in the \( L^2 \)-norm.

43. (Banach-Steinhaus Theorem) Suppose that \( \{ \varphi_n : X \to \mathbb{C} | n \in \mathbb{N} \} \) is a set of linear functionals on a Banach space \( X \) such that the set of complex numbers \( \{ \varphi_n(x) | n \in \mathbb{N} \} \) is bounded for each \( x \in X \). Then \( \{ ||\varphi_n|| | n \in \mathbb{N} \} \) is bounded.

44. Let \( \mathcal{H} \) be a Hilbert space, \( D \subset \mathcal{H} \) is a dense subset, and \( \{ x_n \} \) is a sequence of vectors in \( \mathcal{H} \). Then \( x_n \rightharpoonup x \) if and only if (a) \( \exists M < \infty \) with \( \| x_n \| \leq M, \forall n \geq 1 \), and (b) \( \langle x_n, y \rangle \to \langle x, y \rangle, \forall y \in D \).
45. Let $\mathcal{H}$ be a Hilbert space. (a) If $x_n \to x$, then $\|x\| \leq \lim \inf_{n \to \infty} \|x_n\|$; (b) If $x_n \to x$ and $\lim_{n \to \infty} \|x_n\| = \|x\|$, then $x_n \to x$.

46. (Banach-Alaoglu Theorem) The closed unit ball of a Hilbert space is weakly compact. (A set is weakly precompact if and only if it is bounded.)

47. A function $f : K \to \mathbb{R}$ on a weakly closed set $K$ is said to be weakly sequentially lower semicontinuous if $f(x) \leq \lim \inf_{n \to \infty} f(x_n)$ for every sequence $(x_n)$ in $K$ such that $x_n \to x$.

48. Suppose that $f : K \to \mathbb{R}$ is a weakly lower semicontinuous function on a weakly closed, bounded subset $K$ of a Hilbert space. Then $f$ is bounded form below and attains its infimum.

49. Let $f : C \to \mathbb{R}$ be a real-valued function on a convex subset $C$ of a real or complex linear space. Then $f$ is convex if $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$ for all $x, y \in C$ and $0 \leq t \leq 1$.

50. (Mazur’s Theorem) If $\{x_n\}$ converges weakly to $x$ in a Hilbert space, then there is a sequence $\{y_n\}$ of finite convex combinations of $\{x_n\}$ such that $\{y_n\}$ converges strongly to $x$.

51. Suppose that $f : C \to \mathbb{R}$ is a strongly lower semicontinuous convex function on a strongly closed, convex, bounded subset $C$ of a Hilbert space. Then $f$ is bounded from below and attains its infimum. If $f$ is strictly convex, then the minimizer is unique.

52. The resolvent set of an operator $A \in \mathcal{B}(\mathcal{H})$, denoted by $\rho(A)$, is the set of complex numbers $\lambda$ such that $(A - \lambda I) : \mathcal{H} \to \mathcal{H}$ is one-to-one and onto. The spectrum of $A$, denoted by $\sigma(A)$, is the complement of the resolvent set in $\mathbb{C}$, meaning that $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

53. Suppose that $A$ is a bounded linear operator on a Hilbert space $\mathcal{H}$.

(a) The point spectrum of $A$ consists of all $\lambda \in \sigma(A)$ such that $A - \lambda I$ is not one-to-one. In this case $\lambda$ is called an eigenvalue of $A$.

(b) The continuous spectrum of $A$ consists of all $\lambda \in \sigma(A)$ such that $A - \lambda I$ is one-to-one but not onto, and ran $(A - \lambda I)$ is dense in $\mathcal{H}$.

(c) The residual spectrum of $A$ consists of all $\lambda \in \sigma(A)$ such that $A - \lambda I$ is one-to-one but not onto, and ran $(A - \lambda I)$ is not dense in $\mathcal{H}$.

54. An operator-valued function $F : \Omega \to \mathcal{B}(\mathcal{H})$, defined on an open subset $\Omega$ of the complex plane $\mathbb{C}$, is said to be analytic at $z_0 \in \Omega$ if there are operators $F_n \in \mathcal{B}(\mathcal{H})$ and a $\delta > 0$ such that $F(z) = \sum_{n=0}^{\infty} (z - z_0)^n F_n$, where the power series on the right-hand side converges with respect to the operator norm on $\mathcal{B}(\mathcal{H})$ in a disc $|z - z_0| < \delta$ for some $\delta > 0$.

55. If $\lambda$ belongs to the resolvent set $\rho(A)$ of a linear operator $A$, then $A - \lambda I$ has an everywhere defined, bounded inverse. The operator $R_\lambda = (\lambda I - A)^{-1}$ is called the resolvent of $A$ at $\lambda$.

56. Let $A \in \mathcal{B}(\mathcal{H})$, then (a) $\rho(A)$ is open, (b) $\{\lambda \in \mathbb{C} : ||\lambda|| > ||A||\} \subset \rho(A)$, and (c) $R_\lambda$ is an operator valued analytic function on $\rho(A)$. As a consequence, $\sigma(A)$ is closed, $\sigma(A) \subset \{\lambda \in \mathbb{C} : ||\lambda|| \leq ||A||\}$, and therefore, $\sigma(A)$ is compact.

57. If $A$ is a bounded linear operator, then $r(A) = \lim_{n \to \infty} ||A^n||^{1/n}$. If $A$ is self-adjoint, then $r(A) = ||A||$. 

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58. The spectrum of a bounded linear operator on a Hilbert space is nonempty.

59. Let \( \mathcal{H} \) be a Hilbert space, for any \( A \in \mathcal{B}(\mathcal{H}) \), a subspace \( \mathcal{M} \subset \mathcal{H} \) is called an **A-invariant subspace** if \( \forall x \in \mathcal{M}, Ax \in \mathcal{M} \).

60. Let \( A \in \mathcal{B}(\mathcal{H}) \) be self-adjoint, and \( \mathcal{M} \) is an \( A \)-invariant subspace, then \( \mathcal{M}^\perp \) is also an \( A \)-invariant subspace.

61. Let \( A \in \mathcal{B}(\mathcal{H}) \), if \( \lambda \) is in the residual spectrum of \( A \), then \( \overline{\lambda} \) is an eigenvalue of \( A^* \).

62. If \( A \in \mathcal{B}(\mathcal{H}) \) is a self-adjoint operator on a Hilbert space, then the spectrum of \( A \) is real and is contained in the interval \([-\|A\|,\|A\|]\).

63. If \( A \in \mathcal{B}(\mathcal{H}) \) satisfying \( A^* = A \), then the residual spectrum of \( A \) is empty.

64. Let \( A \in \mathcal{B}(\mathcal{H}) \) satisfying \( A \) is self-adjoint and compact. If \( \lambda \neq 0 \) is an eigenvalue of \( A \), then \( \lambda \) has finite multiplicity. If \( A \) has countably many non-zero eigenvalues, then zero is the only accumulative point.

65. (Spectral Theorem) Let \( A : \mathcal{H} \to \mathcal{H} \) be a compact, self-adjoint operator on a Hilbert space \( \mathcal{H} \). There is an orthonormal basis of \( \mathcal{H} \) consisting of eigenvectors of \( A \). The nonzero eigenvalues of \( A \) form a finite or countably infinite set \( \{\lambda_k\} \) of real numbers, and \( A = \sum_k \lambda_k P_k \), where \( P_k \) is the orthogonal projection onto the finite-dimensional eigenspace of eigenvectors with eigenvalue \( \lambda_k \). If the number of nonzero eigenvalues is countably infinite, then the series converges to \( A \) in the operator norm.

66. Let \( E \) be a subset of an infinite-dimensional, separable Hilbert space \( \mathcal{H} \). (a) If \( E \) is precompact, then for every orthonormal set \( \{e_n|n \in \mathbb{N}\} \) and every \( \epsilon > 0 \), there is an \( N \) such that \( \sum_{n=N+1}^{\infty} |\langle e_n, x \rangle|^2 < \epsilon \) for all \( x \in E \). (b) If \( E \) is bounded and there is an orthonormal basis \( \{e_n\} \) of \( \mathcal{H} \) with the property that for every \( \epsilon > 0 \) there is an \( N \) such that \( \sum_{n=N+1}^{\infty} |\langle e_n, x \rangle|^2 < \epsilon \) for all \( x \in E \), then \( E \) is precompact.

67. A bounded linear operator \( A \) on a separable Hilbert space \( \mathcal{H} \) is **Hilbert-Schmidt** if there is an orthonormal basis \( \{e_n|n \in \mathbb{N}\} \) such that \( \sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty \). If \( A \) is a Hilbert-Schmidt operator, then \( \|A\|_{HS} = \sqrt{\sum_{n=1}^{\infty} \|Ae_n\|^2} \) is called the **Hilbert-Schmidt norm** of \( A \).

68. A Hilbert-Schmidt operator is compact.

69. A bounded linear operator on a Hilbert space is compact if and only if it maps weakly convergent sequences into strongly convergent sequences.

70. (Spectral Mapping) Let \( A \) be compact and self-adjoint. If \( f : \sigma(A) \to \mathbb{C} \) is continuous, then \( \sigma(f(A)) = f(\sigma(A)) \).

71. A function \( f : (a, b) \to X \) from an open interval \( (a, b) \) into a Banach space \( X \) is **differentiable** at \( a < t < b \), with derivative \( f'(t) \in X \), if the following limit exists in \( X \): \( f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \).

The function \( f \) is differentiable in \( (a, b) \) if it is differentiable at each point in \( (a, b) \), and continuously differentiable in \( (a, b) \) if \( f' : (a, b) \to X \) is continuous.

72. If \( f : (a, b) \to X \) is differentiable in \( (a, b) \) and \( f' = 0 \), then \( f \) is a constant function.
73. (Fundamental Theorem of Calculus) Suppose that $X$ is a Banach space. (a) If $f : [a, b] \to X$ is continuous, then $F(t) = \int_a^t f(s)ds$ is continuously differentiable in $(a, b)$ and $F' = f$. (b) If $f$ is continuously differentiable in an open interval containing $[a, b]$, then $f(b) - f(a) = \int_a^b f'(t)dt$.

74. (Mean Value) If $f$ is continuously differentiable in an open interval that contains the closed, bounded interval $[a, b]$, with values in a Banach space, then $\|f(b) - f(a)\| \leq M(b - a)$ where $M = \sup_{a \leq t \leq b} \|f'(t)\|$.

75. A map $f : U \subset X \to Y$ whose domain $U$ is an open subset of a Banach space $X$ and whose range is a Banach space $Y$ is **differentiable** at $x \in U$ if there is a bounded linear map $A : X \to Y$ such that $\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0$ (i.e., $f(x + h) = f(x) + Ah + o(h)$). The linear operator $A$ is called the **Fréchet derivative** of $f$ at $x \in U$.

76. (Chain Rule) Suppose that $X, Y, Z$ are Banach spaces, and $f : U \subset X \to Y$, $g : V \subset Y \to Z$ where $U$ and $V$ are open subsets of $X$ and $Y$, respectively. If $f$ is differentiable at $x \in U$ and $g$ is differentiable at $f(x) \in V$, then $g \circ f$ is differentiable at $x$ and $(g \circ f)'(x) = g'(f(x))f'(x)$.

77. Let $X$ and $Y$ be Banach spaces with $f : U \subset X \to Y$ and $U$ open, the **directional derivative** of $f$ at $x \in U$ in the direction of $h \in X$ is $\delta f(x; h) = \lim_{t \to 0} \frac{f(tx + th) - f(tx)}{t}$ if this limit exists in $Y$. If this limit exists for all $h \in X$ and $f_G'(x) : X \to Y$ defined by $f_G'(x)h = \delta f(x; h)$ is linear, then we say that $f$ is **Gâteaux differentiable** at $x$ and $f_G'(x)$ is the **Gâteaux derivative** of $f$ at $x$.

78. Let $f : X \to Y$ be Gâteaux differentiable for all $x \in U \subset X$ with $U$ open. If $x, y \in U$ and the line segment $\{tx + (1-t)y | 0 \leq t \leq 1\} \subset U$, then $\|f(x) - f(y)\| \leq M\|x - y\|$ where $M = \sup_{0 \leq t \leq 1} \|f_G''(tx + (1-t)y)\|$.

79. Let $f : U \subset X \to Y$ be Gâteaux differentiable, and $U$ is a convex open set. If the Gâteaux derivative: $f_G' : U \to \mathcal{B}(X, Y)$ is continuous at $x \in U$, then $f$ is Fréchet differentiable at $x$ and $f'(x) = f_G'(x)$.

80. (Inverse Function Theorem) Let $X$ and $Y$ be Banach spaces, let $f : U \subset X \to Y$ be differentiable on $U$. If $f$ is continuously differentiable on $U$ and $f'(x)$ has a bounded inverse at $x \in U$, then there are open sets $x \in V \subset U$ and $W \subset Y$ with $f(x) \in W$ such that $f : V \to W$ is one-to-one and onto. Moreover, $f^{-1} : W \to V$ is continuously differentiable at $f(x)$ with $(f^{-1})'(f(x)) = (f'(x))^{-1}$. 
