MAT 206 RECAP

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1. Let $\mu$ be a measure $\mu : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ such that (i) if $E \subset F$, then $\mu(E) \leq \mu(F)$; (ii) if $E_1, E_2, \cdots$ is a countable collection of disjoint sets, then $\mu(\bigcup E_j) = \sum \mu(E_j)$; (iii) if $E \in \mathcal{P}(\mathbb{R})$ and $F = \{x - \tau : x \in E\}$ for any $\tau \in \mathbb{R}$, have $\mu(E) = \mu(F)$; and (iv) $\mu([0, 1]) = 1$. Then there is no such mapping.

2. Let $X$ be a nonempty set, we call a collection $\mathcal{A}$ of subsets of $X$ an algebra provided: (i) $\emptyset \in \mathcal{A}$; (ii) if $E, F \in \mathcal{A}$, then $E \cup F \in \mathcal{A}$; and (iii) if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.

3. Let $X$ be a nonempty set, we call a collection $\mathcal{A}$ of subsets of $X$ an $\sigma$-algebra provided: (i) $\emptyset \in \mathcal{A}$; (ii) if $\{E_j\}_{j=1}^\infty \subset \mathcal{A}$, then $\bigcup_{j=1}^\infty E_j \in \mathcal{A}$; and (iii) if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.

4. If $\mathcal{A}$ is an algebra and it is closed under countable disjoint unions, then $\mathcal{A}$ is a $\sigma$-algebra.

5. The intersection of any collection of $\sigma$-algebras is a $\sigma$-algebra.

6. Suppose $X$ is a metric space, let $\mathcal{B}_X$ be the $\sigma$-algebra generated by the open sets of $X$, we call $\mathcal{B}_X$ the Borel $\sigma$-algebra and its elements Borel sets.

7. A countable intersection of open sets is called a $G_\delta$ set; a countable union of closed sets is called an $F_\sigma$ set.

8. We call a collection of subsets of $X$ an elementary family if (i) $\emptyset \in \mathcal{E}$; (ii) if $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$; and (iii) if $E \in \mathcal{E}$, then $E^c$ is a finite disjoint union of sets in $\mathcal{E}$.

9. If $\mathcal{E}$ is an elementary family, then the collection of finite disjoint unions of $\mathcal{E}$ is an algebra.

10. Let $X$ be a nonempty subset and $\mathcal{M}$ a $\sigma$-algebra of sets of $X$, we say a function $\mu : \mathcal{M} \to [0, \infty]$ is a measure if (i) $\mu(\emptyset) = 0$; (ii) if $\{E_j\}_{j=1}^\infty$ is a collection of disjoint sets in $\mathcal{M}$, then $\mu\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \mu(E_j)$. We call the triple $(X, \mathcal{M}, \mu)$ a measure space and sets in $\mathcal{M}$ measurable sets.

11. In a measure space $(X, \mathcal{M}, \mu)$, we say a measure $\mu$ is finite if $\mu(X) < \infty$. We say $\mu$ is $\sigma$-finite if there exists $\{E_j\}_{j=1}^\infty$ such that $X = \bigcup E_j$ and $\mu(E_j) < \infty$. If for each $E \in \mathcal{M}$ such that $\mu(E) = \infty$, there exists $\emptyset \subsetneq F \subsetneq E$ such that $\mu(F) < \infty$, then we say $\mu$ is semifinite.

12. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space, then
   (a) if $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) \leq \mu(F)$;
   (b) if $\{E_j\}_{j=1}^\infty \subset \mathcal{M}$, then $\mu(\bigcup E_j) \leq \sum \mu(E_j)$;
   (c) if $E_1 \subset E_2 \subset E_3 \subset \cdots$ are measurable, then $\mu(\bigcup E_j) = \lim \mu(E_j)$;
   (d) if $E_1 \supset E_2 \supset E_3 \supset \cdots$ are measurable and $\mu(E_1) < \infty$, then $\mu(\bigcap E_j) = \lim \mu(E_j)$.

13. We say $(X, \mathcal{M}, \mu)$ is complete if whenever $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $N \subset E$, we have $N \in \mathcal{M}$.

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14. Suppose \((X, \mathcal{M}, \mu)\) is a measure space, let \(\mathcal{N} = \{K : K \subset E \text{ with } E \in \mathcal{M} \text{ and } \mu(E) = 0\}\); \(\overline{\mathcal{M}} = \{E \cup F : E \subset \mathcal{M}, F \subset N \text{ for some } N \in \mathcal{N}\}\). There is a unique extension of \(\mathcal{M}\) to \(\overline{\mathcal{M}}\).

15. Let \(X\) be a nonempty set, an outer measure on \(X\) is a function \(\mu^* : \mathcal{P}(X) \to [0, \infty]\) such that (i) \(\mu^*(\emptyset) = 0\); (ii) \(\mu^*(A) \leq \mu^*(B)\) if \(A \subset B\); and (iii) \(\mu^*(\bigcup A_j) \leq \sum \mu^*(A_j)\).

16. Let \(\mathcal{E} \subset \mathcal{P}(X)\) and \(\rho : \mathcal{E} \to [0, \infty]\) be such that (i) \(\emptyset, X \in \mathcal{E}\) and (ii) \(\rho(\emptyset) = 0\). Then the function \(\mu^* : \mathcal{P}(X) \to [0, \infty]\) defined by \(\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) : A \subset \bigcup_{j=1}^{\infty} E_j \right\}\) is an outer measure.

17. Suppose that \(\mu^*\) is an outer measure on \(X\), we say that a set \(A \subset X\) is \(\mu^*\)-measurable if \(\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)\) for all \(E \subset X\).

18. Suppose that \(\mu^*\) is an outer measure on a nonempty set \(X\). Then the collection \(\mathcal{M}\) of \(\mu^*\)-measurable sets is a \(\sigma\)-algebra and the restriction \(\mu^*|_E\) is a complete measure.

19. Suppose that \(\mathcal{A} \subset \mathcal{P}(X)\) is an algebra, we say \(\mu_0 : \mathcal{A} \to [0, \infty]\) is a premeasure if (i) \(\mu_0(\emptyset) = 0\); and (ii) \(\mu_0(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu_0(A_j)\) whenever disjoint \(\{A_j\} \subset \mathcal{A}\) and \(\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}\).

20. If \(\mathcal{A}\) is an algebra and \(\mu_0\) is a premeasure on \(\mathcal{A}\). Suppose that \(\mu^*\) is the outer measure induced by \(\mu_0 : \mathcal{A} \to [0, \infty]\). Then (i) \(\mu^*|_{\mathcal{A}} = \mu_0\), and (ii) every set in \(\mathcal{A}\) is \(\mu^*\)-measurable.

21. Suppose that \(\mu_0\) is a premeasure on an algebra \(\mathcal{A}\), and \(\mathcal{M}\) is the \(\sigma\)-algebra generated by \(\mathcal{A}\). Let \(\mu^*\) be the outer measure induced by \(\mu_0\), and \(\mu = \mu^*|_{\mathcal{M}}\). If \(\nu\) is a measure on \(\mathcal{M}\) such that \(\nu|_{\mathcal{A}} = \mu|_{\mathcal{A}}\), then \(\nu(E) \leq \mu(E)\) for all \(E \in \mathcal{M}\). If \(\mu(E) < \infty\), then \(\nu(E) = \mu(E)\). If \(\mu\) is \(\sigma\)-finite, then \(\nu = \mu\).

22. Let \(\mathcal{E}\) be the collection of \(h\)-intervals, i.e., sets of the form \((a, b]\) for \(-\infty \leq a < b < \infty\) or \((a, \infty]\) for \(-\infty \leq a \leq b < \infty\) or \((a, \infty)\) for \(-\infty < a < b < \infty\) or \((a, \infty)\) for \(-\infty < a < b < \infty\). Let \(F : \mathbb{R} \to \mathbb{R}\) be increasing and right continuous. Let \(\mathcal{A}\) be the collection of finite disjoint unions of sets in \(\mathcal{E}\). Define \(\mu_0 : \mathcal{A} \to [0, \infty]\) via \(\mu_0(\bigcup (a_j, b_j]) = \sum F(b_j) - F(a_j)\). Then \(\mathcal{A}\) is an algebra and \(\mu_0\) a premeasure on \(\mathcal{A}\). The induced measure is called Lebesgue-Stieltjes measure associated with \(F\).

23. Suppose that \(\mu\) is the Lebesgue measure associated with \(F : \mathbb{R} \to \mathbb{R}\). Then (i) \(\mu\{(a, b]\} = F(b) - F(a]\); (ii) \(\mu((a, b)) = F(b) - F(a)\); (iii) \(\mu([a, b]) = F(b) - F(a)\); (iv) \(\mu((a, b)) = F(b) - F(a)\).

24. If \(E\) is measurable, then \(\mu(E) = \inf \{\sum \mu((a_j, b_j)) : E \subset \bigcup (a_j, b_j)\}\).

25. If \(E\) is measurable, then
\[
\mu(E) = \inf \{\mu(U) : E \subset U \text{ and } U \text{ is open}\} = \sup \{\mu(K) : K \subset E \text{ and } K \text{ is compact}\}
\]

26. If \(E\) is measurable, \(\mu(E) < \infty\) and \(\epsilon > 0\), then there exists a set \(A\) which is a finite union of open intervals such that \(\mu(E \Delta A) < \epsilon\).

27. Lebesgue measure, \(m\), is the Lebesgue-Stieltjes measure associated with \(F(x) = x\). \(m\) is translation invariant and \(m(rE) = |r| \cdot m(E)\).

28. Suppose \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) are measure spaces. We say \(f : X \to Y\) is \((\mathcal{M}, \mathcal{N})\)-measurable if \(f^{-1}(E) \in \mathcal{M}\) whenever \(E \in \mathcal{N}\).

29. The set \(\{f^{-1}(E) | E \in \mathcal{N}\}\) is a \(\sigma\)-algebra assuming \(\mathcal{N}\) is a \(\sigma\)-algebra.
30. \( f : X \to \mathbb{R} \) (or \( \mathbb{C} \)) is measurable if and only if \( f^{-1}((a, \infty)) \in \mathcal{M} \) for all \( a \in \mathbb{R} \).

31. A function \( f : \mathbb{R} \to \mathbb{R} \) is **Borel measurable** if it is \((B_{\mathbb{R}}, B_{\mathbb{R}})\)-measurable. And it is **Lebesgue measurable** if it is \((\mathcal{L}, B_{\mathbb{R}})\)-measurable, where \( \mathcal{L} \) is Lebesgue \( \sigma \)-algebra, the completion of Borel \( \sigma \)-algebra.

32. If \( f_n : \mathbb{R} \to \mathbb{R} (= \mathbb{R} \cup \{\pm \infty\}) \) is a sequence of measurable functions, then \( \limsup f_n \) and \( \liminf f_n \) are measurable.

33. If \((X, \mathcal{M}, \mu)\) is complete, \( f : X \to \mathbb{R} \) is measurable, and \( g : X \to \mathbb{R} \) is such that \( f(x) = g(x) \) \( \mu \)-a.e., then \( g \) is measurable.

34. Let \((X, \mathcal{M}, \mu)\) be a measure space, we say \( \phi : X \to \mathbb{R} \) is a **simple function** if there exists a collection of disjoint measurable sets \( E_1, E_2, \ldots, E_n \) and real numbers \( a_1, a_2, \ldots, a_n \) such that \( \phi(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x) \).

35. Suppose \((X, \mathcal{M}, \mu)\) is a measure space and \( f : X \to [0, \infty] \) is measurable. Then there exists a sequence \( \{\phi_n\} \) of simple functions such that \( 0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f \) and \( \phi_n \to f \) pointwise. If \( f \) is bounded on \( E \in \mathcal{M} \), then \( \sup_{x \in E} |f(x) - \phi_n(x)| \to 0 \) as \( n \to \infty \).

36. Suppose \((X, \mathcal{M}, \mu)\) is a measure space and \( f : X \to \mathbb{C} \) is measurable. Then there exists a sequence \( \{\phi_n\} \) of simple functions such that \( 0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |f| \) and \( \phi_n \to f \) pointwise. If \( f \) is bounded on \( E \in \mathcal{M} \), then \( \sup_{x \in E} |f(x) - \phi_n(x)| \to 0 \) as \( n \to \infty \).

37. If \( c \geq 0 \) and \( \phi \) and \( \psi \) are simple, then (i) \( \int \phi = \int c \phi \); (ii) \( \int \phi + \psi = \int \phi + \int \psi \); (iii) If \( \phi \leq \psi \), then \( \int \phi \leq \int \psi \); (iv) The map \( A \to \int_A \phi \equiv \int \phi \chi_A \) is a measure.

38. (Monotone Convergence Theorem) If \( \{f_n\} \) is a sequence of measurable functions in \( L^+ = \{f : X \to [0, \infty] \mid f \text{ is measurable}\} \) and \( 0 \leq f_1 \leq f_2 \leq \cdots \leq f \) and \( \lim_{n \to \infty} f_n = f \), then \( \lim_{n \to \infty} \int f_n = \int \lim_{n \to \infty} f_n = \int f \).

39. If \( f \in L^+ \), then \( \int f = 0 \) if and only if \( \mu(\{x : f(x) > 0\}) = 0 \).

40. (Fatou’s Lemma) Suppose \( \{f_n\} \subset L^+ \), then \( \int \liminf f_n \leq \liminf \int f_n \).

41. If \( f \in L^+ \) and \( \int f < \infty \), then \( m(\{x : f(x) = \infty\}) = 0 \) and \( \{x : f(x) > 0\} \) is \( \sigma \)-finite.

42. (Dominated Convergence Theorem) If \( \{f_n\} \) is a sequence of measurable functions such that (i) \( \lim f_n \to f \) a.e., and (ii) there exists \( g \in L^1(X) \) with \( |f_n| \leq g \), then \( f \in L^1(X) \) and \( \int f_n \to \int f \).

43. Suppose \( f : X \to \mathbb{C} \) is in \( L^1(X) \), for every \( \epsilon > 0 \), there exists a simple function \( \phi \) such that \( \int |f - \phi| < \epsilon \).

44. Suppose \( \mu \) is a Lebesgue-Stieltjes measure and that \( f : X \to \mathbb{C} \) is in \( L^1(X) \). Then for any \( \epsilon > 0 \), there exists a simple function \( \phi \) of the form \( \phi = \sum_{j=1}^{n} a_j \chi_{E_j} \) with \( \{E_j\} \) open intervals such that \( \int |f - \phi| < \epsilon \).

45. Suppose \( f : X \times [a, b] \to \mathbb{C} \) and \( f(\cdot, t) \) is measurable for each \( t \). Suppose also that \( f(x, \cdot) \) is continuous for each \( x \), and there exists \( g \in L^1(X) \) such that \( |f(x, t)| < g(x) \) for all \( x \) and \( t \), then \( F(t) = \int f(x, t)dx \) is continuous.
46. Suppose \( f : X \times [a,b] \to \mathbb{C} \) and \( f(\cdot, t) \) is integrable for each \( t \). Suppose also that \( \frac{\partial f}{\partial t} \) exists for each \( x \) and that there exists \( g \in L^1(X) \) such that \( \left| \frac{\partial f}{\partial t}(x,t) \right| < g(x) \) for all \( x \) and \( t \), then
\[
\frac{d}{dt} \int f(x,t)dx = \int \frac{\partial f}{\partial t}(x,t)dx.
\]

47. We say a sequence \( \{f_n\} \) of measurable complex-valued functions on \((X, \mathcal{M}, \mu)\) is **Cauchy in measure** if for every \( \epsilon > 0 \), \( \mu\left( \left\{ x : |f_n(x) - f_m(x)| \geq \epsilon \right\} \right) \to 0 \) as \( m, n \to \infty \), and that \( \{f_n\} \) **converges in measure** to \( f \) if for every \( \epsilon > 0 \), \( \mu\left( \left\{ x : |f_n(x) - f(x)| \geq \epsilon \right\} \right) \to 0 \) as \( n \to \infty \).

48. If \( f_n \to f \) in \( L^1(X) \), then \( f_n \to f \) in measure.

49. If \( f_n \to f \) in measure, then \( \{f_n\} \) is Cauchy in measure.

50. If \( \{f_n\} \) is Cauchy in measure, then there exists a subsequence \( \{f_{n_j}\} \) of \( \{f_n\} \) and a measurable function \( f \) such that \( f_{n_j} \to f \) a.e..

51. (Egoroff’s Theorem) Suppose that \( \mu(X) < \infty \), and \( f_1, f_2, \cdots \) and \( f \) are measurable complex-valued functions on \( X \) such that \( f_n \to f \) a.e.. Then for every \( \epsilon > 0 \), there exists \( E \subset X \) such that \( \mu(E) < \epsilon \) and \( f_n \to f \) uniformly on \( E^c \).

52. (Lusin’s Theorem) Suppose \( f : [a,b] \to \mathbb{C} \) is measurable. Then for all \( \epsilon > 0 \), there exists a set \( E \subset [a,b] \) such that \( m([a,b] \setminus E) < \epsilon \) and \( f|_E \) is continuous.

53. Let \( (X, \mathcal{M}, \mu) \) and \( (Y, \mathcal{N}, \nu) \) be measure spaces. We have product \( \sigma \)-algebra \( \mathcal{M} \otimes \mathcal{N} \subset \mathcal{P}(X \times Y) \) and product \( \mu \times \nu : \mathcal{M} \otimes \mathcal{N} \to [0, \infty] \) with \( \mu \times \nu(A \times B) = \mu(A)\nu(B) \) for all \( A \in \mathcal{M} \) and \( B \in \mathcal{N} \). We call a set of the form \( A \times B \) with \( A \in \mathcal{M} \) and \( B \in \mathcal{N} \) a **(measurable) rectangle**. If \( E \subset X \times Y \), for \( x \in X \) and \( y \in Y \) we define the \( x \)-**section** \( E_x \) and the \( y \)-**section** \( E_y \) of \( E \) by \( E_x = \{ y \in Y : (x, y) \in E \} \), and \( E_y = \{ x \in X : (x, y) \in E \} \).

54. \( B_{\mathbb{R}^n} \simeq B_{\mathbb{R}} \otimes \cdots \otimes B_{\mathbb{R}} = \mathcal{L} \otimes \cdots \otimes \mathcal{L} = \mathcal{L}^p \)

55. Suppose \( E \in \mathcal{M} \otimes \mathcal{N} \) and \( f : X \times Y \to \mathbb{C} \) is measurable, then \( E_x \in \mathcal{N} \) for all \( x \), and \( E^y \in \mathcal{M} \) for all \( y \).

56. A **monotone class** on a space \( X \) is a subset \( \mathcal{C} \) of \( \mathcal{P}(X) \) that is closed under countable increasing unions and countable decreasing intersections. \( \sigma \)-algebra is a monotone class, so for any \( \mathcal{E} \subset \mathcal{P}(X) \) there is a unique smallest monotone class containing \( \mathcal{E} \), called monotone class \( \mathcal{C} \) generated by \( \mathcal{E} \).

57. (The Monotone Class Lemma) If \( \mathcal{A} \) is an algebra of subsets of \( X \), then the monotone class \( \mathcal{C} \) generated by \( \mathcal{A} \) coincides with the \( \sigma \)-algebra \( \mathcal{M} \) generated by \( \mathcal{A} \).

58. (The Fubini-Tonelli Theorem) Suppose that \( (X, \mathcal{M}, \mu) \) and \( (Y, \mathcal{N}, \nu) \) are \( \sigma \)-finite measure spaces.

(a) (Tonelli) If \( f \in L^+(X \times Y) \), then the functions \( g(x) = \int f_x d\nu \) and \( h(y) = \int f^y d\mu \) are in \( L^+(X) \) and \( L^+(Y) \), respectively, and \( \int f d(\mu \times \nu) = \int \int f(x,y) d\nu(y) d\mu(x) = \int \int f(x,y) d\mu(x) d\nu(y) \).

(b) (Fubini) If \( f \in L^1(\mu \times \nu) \), then \( f_x \in L^1(\nu) \) for a.e. \( x \in X \), \( f^y \in L^1(\mu) \) for a.e. \( y \in Y \), the a.e.-defined functions \( g(x) = \int f_x d\nu \) and \( h(x) = \int f^x d\mu \) are in \( L^1(\mu) \) and \( L^1(\nu) \), respectively, and \( \int f d(\mu \times \nu) = \int \int f(x,y) d\nu(y) d\mu(x) = \int \int f(x,y) d\mu(x) d\nu(y) \) holds.
59. The Lebesgue measure on \( \mathbb{R}^n \) is translation invariant.

60. Suppose \( E \subset \mathbb{R}^n \) is Lebesgue measurable, then (i) \( m(E) = \sup \{ m(K) | K \subset E \text{ and } K \text{ is compact} \} \); (ii) \( m(E) = \inf \{ m(U) | E \subset U \text{ and } U \text{ is open} \} \); (iii) If \( m(E) < \infty \), then for all \( \epsilon > 0 \), there exists a finite disjoint collection \( \{ R_j \}_{i=1}^N \) of sets whose sides are open intervals such that \( m(E \triangle (\cup R_j)) < \epsilon \).

61. Suppose that \( T : \mathbb{R}^n \to \mathbb{R}^n \) is an invertible linear mapping, if \( f : \mathbb{R}^n \to \mathbb{R}^n \) is measurable, then \( f \circ T \) is measurable. If \( f \geq 0 \) or \( f \in L^1(\mathbb{R}^n) \), then \( \int_{\mathbb{R}^n} f(x) = |\det T| \int_{\mathbb{R}^n} f(T(x))dx \).

62. Suppose that \( \Omega \subset \mathbb{R}^n \) is open and that \( T : \Omega \to \mathbb{R}^n \) is 1-1 and continuously differentiable. If \( f : \mathbb{R}^n \to \mathbb{R} \) is Lebesgue measurable, then \( f \circ T : \Omega \mathbb{R} \) is as well. If \( f \geq 0 \) or \( f \in L^1(\mathbb{R}^n) \), then \( \int_{T(\Omega)} f = \int_\Omega f(T(x))|dT(x)|dx \).

63. A signed measure on \((X, \mathcal{M})\) is a function \( \nu : \mathcal{M} \to [\mathbb{R}, \mathbb{R}] \) such that (i) \( \nu(\varnothing) = 0 \), (ii) \( \nu \) assumes at most one of the values \( \pm \infty \), and (iii) if \( \{E_j\} \) is a sequence of disjoint sets in \( \mathcal{M} \), then \( \nu(\cup E_j) = \sum \nu(E_j) \) where the latter sum converges absolutely if \( \nu(\cup E_j) \) is finite.

64. If \( \nu \) is a signed measure on \((X, \mathcal{M})\), a set \( E \in \mathcal{M} \) is called positive (resp. negative, null) for \( \nu \) if \( \nu(F) \geq 0 \) (resp. \( \nu(F) \leq 0, \nu(F) = 0 \)) for all \( F \in \mathcal{M} \) such that \( F \subset E \).

65. (Hahn Decomposition Theorem) If \( \nu \) is a signed measure on \((X, \mathcal{M})\), there exist a positive set \( P \) and a negative set \( N \) for \( \nu \) such that \( P \cup N = X \) and \( P \cap N = \varnothing \). If \( P', N' \) is another such pair, then \( P \triangle P' = (N \triangle N') \) is null for \( \nu \).

66. Two signed measures \( \mu \) and \( \nu \) on \((X, \mathcal{M})\) are mutually singular, if there exist \( E, F \in \mathcal{M} \) such that \( E \cap F = \varnothing, E \cup F = X, E \) is null for \( \mu \), and \( F \) is null for \( \nu \), denoted by \( \mu \perp \nu \).

67. (Jordan Decomposition Theorem) If \( \nu \) is a signed measure, there exist unique positive measures \( \nu^+ \) and \( \nu^- \) such that \( \nu = \nu^+ - \nu^- \) and \( \nu^+ \perp \nu^- \).

68. Suppose \( \nu \) is a signed measure and \( \mu \) is a positive measure on \((X, \mathcal{M})\). We say that \( \nu \) is absolutely continuous with respect to \( \mu \) and write \( \nu \ll \mu \) if \( \nu(E) = 0 \) for every \( E \in \mathcal{M} \) for which \( \mu(E) = 0 \).

69. Let \( \nu \) be a finite signed measure and \( \mu \) a positive measure on \((X, \mathcal{M})\). Then \( \nu \ll \mu \) if and only if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |\nu(E)| < \epsilon \) whenever \( \mu(E) < \delta \).

70. If \( f \in L^1(\mu) \), for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |\int_E f d\mu| < \epsilon \) whenever \( \mu(E) < \delta \).

71. Suppose that \( \nu \) and \( \mu \) are finite measures on \((X, \mathcal{M})\), either \( \nu \perp \mu \), or there exists \( \epsilon > 0 \) and \( E \in \mathcal{M} \) such that \( \mu(E) > 0 \) and \( \nu \geq \epsilon \mu \) on \( E \) (that is, \( E \) is a positive set for \( \nu - \epsilon \mu \)).

72. (Lebesgue-Radon-Nikodym Theorem) Let \( \nu \) be a \( \sigma \)-finite signed measure and \( \mu \) a \( \sigma \)-finite positive measure on \((X, \mathcal{M})\). There exist unique \( \sigma \)-finite signed measures \( \lambda, \rho \) on \((X, \mathcal{M})\) such that \( \lambda \perp \mu, \rho \ll \mu \), and \( \nu = \lambda + \rho \). Moreover, there is an extended \( \mu \)-integrable function \( f : X \to \mathbb{R} \) such that \( d\rho = fd\mu \), and any two such functions are equal \( \mu \text{-a.e.} \).

73. We say \( f : \mathbb{R}^n \to \mathbb{R} \) is locally integrable if \( \int_K |f| < \infty \), where \( K \) is a bounded measurable set. We call the set of such functions \( L^1_{loc}(\mathbb{R}^n) \).

74. We denote \( A_r[f](x) := \frac{1}{m(B_r(x))} \int_{B_r(x)} f(u)du \), then \( A_r[f] \) is continuous in \( r \) and \( x \).
75. The **Hardy-Littlewood maximal function** \( H[f] \) is defined for \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) via \( H[f](x) = \sup_{r>0} A_r([f])(x) \).

76. If \( H[f] \in L^1 \), then \( f = 0 \) a.e.

77. Suppose that \( \mathcal{C} \) is a collection of open balls in \( \mathbb{R}^n \) and let \( U = \bigcup_{B \in \mathcal{C}} B \). If \( c < mU \), then there exists disjoint balls \( B_1, B_2, \ldots, B_k \) in \( \mathcal{C} \) such that \( \sum_{j=1}^k m(B_j) > 3^{-n}c \).

78. (The Maximal Theorem) There is a constant \( C > 0 \) such that for all \( f \in L^1 \) and all \( \alpha > 0 \),
\[
m(\{x: H[f](x) > \alpha \}) \leq \frac{C \int |f(x)| dx}{\alpha}.
\]

79. If \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), then \( \lim_{r \to 0} A_r[f](x) = f(x) \) for almost all \( x \in \mathbb{R}^n \).

80. If \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), then we call the set \( L_f = \left\{ x: \lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(x) - f(y)| dy = 0 \right\} \) the **Lebesgue set** of \( f \).

81. We say a collection \( \{E_r\}_{r>0} \) of sets **shrinks nicely** to \( x \) if (i) \( E_r \subset B_r(x) \) for all \( r > 0 \), and (ii) there exists \( \alpha > 0 \) such that \( m(E_r) > \alpha m(B_r(x)) \) for all \( r > 0 \).

82. If \( x \in L_f \) and \( \{E_r\}_{r>0} \) shrinks nicely to \( x \), then \( \lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x) \).

83. Suppose \( F: \mathbb{R} \to \mathbb{R} \) is increasing and that \( G \) is defined by \( G(x) = F(x+) = \lim_{y \to x^+} F(y) \), then (i) the set of discontinuities of \( F \) is countable, and (ii) \( F \) and \( G \) are differentiable almost everywhere and \( F' = G' \) almost everywhere.

84. We say a signed or complex measure \( \nu \) on \( \mathbb{R}^n \) is **regular** if (i) \( \nu(K) < \infty \) whenever \( K \) is compact, and (ii) \( \nu(E) = \inf\{\nu(U) | E \subset U, U \text{ open}\} \).

85. If \( \nu \perp m \) and \( \lambda \) is regular, then \( \lim_{r \to 0} \frac{\lambda(E_r)}{m(E_r)} = 0 \) whenever \( \{E_r\} \) shrinks nicely to \( x \).

86. If \( F: \mathbb{R} \to \mathbb{C} \) and \( x \in \mathbb{R} \), we define \( T[F](x) = \sup \left\{ \sum_{j=1}^n |F(x_j - x_{j-1})|: n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x \right\} \) to be the **total variation function** of \( F \). If \( T[f](\infty) = \lim T[f](x) \) is finite, we say \( F \) is of **bounded variation** on \( \mathbb{R} \). We denote the space of all such \( F \) by \( BV \). **Normalized bounded variation** functions form Banach space \( NBV = \{ F \in BV | F \text{ is right continuous and } F(-\infty) = 0 \} \).

87. If \( F \in BV \) is real-valued, then \( T_F + F \) and \( T_F - F \) are increasing.

88. If \( F: \mathbb{R} \to \mathbb{R} \), then \( F \in BV \) if and only if \( F \) is the difference of two bounded increasing functions.

89. If \( F \in BV \), then \( F(-\infty) = 0 \). If \( F \) is also right continuous, then so is \( T_F \).

90. If \( -\infty < a < b < \infty \) and \( F: [a, b] \to \mathbb{C} \), the following are equivalent:

   (a) \( F \) is absolute continuous on \([a, b]\).

   (b) \( F(x) - F(a) = \int_a^x f(t) dt \) for some \( f \in L^1([a, b], m) \).

   (c) \( F \) is differentiable almost everywhere, on \([a, b] \), \( F' \in L^1([a, b], m) \), and \( F(x) - F(a) = \int_a^x F'(t) dt \).