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Abstract

This is a short course on instability in interfacial fluid mechanics taught by Professor Ian Tice. This course is one of the three series of courses offered in the USC Summer School on Mathematical Fluids organized by Professor Juhi Jang in University of Southern California during May 22 and May 26, 2017.

1 A crash course on basic fluid mechanics

1.1 Equations of motion for a fluid

For a fluid evolving due to purely mechanical effects, i.e., neglecting thermodynamics, we can specify the state of the fluid with the following for \( t \geq 0 \).

1.1.1 Notations

Let \( \Omega(t) \subset \mathbb{R}^3 \) be an open set in which the fluid resides. We refer this set as fluid domain. We have following functions describing features of fluid:

- \( \rho(\cdot,t) : \Omega(t) \to (0, \infty) \) denotes the density of fluid,
- \( u(\cdot,t) : \Omega(t) \to \mathbb{R}^3 \) denotes the velocity of fluid,
- \( p(\cdot,t) : \Omega(t) \to \mathbb{R} \) denotes the pressure of fluid,
- \( S(\cdot,t) : \Omega(t) \to \text{Sym}_3 = \{ M \in \mathbb{R}^{3 \times 3} \mid M = M^T \} \) denotes the stress tensor, and
- \( f(\cdot,t) : \Omega(t) \to \mathbb{R}^3 \) denotes the external force on fluid.

In these notes, we use Einstein’s summation convention. The \( k \)-th partial derivative of \( \varphi \) will be denoted by \( \varphi_{,k} = \frac{\partial \varphi}{\partial x_k} \). Repeated Latin indices \( i, j, k \), etc., are summed from 1 to 3, and repeated Greek indices \( \alpha, \beta, \gamma \), etc., are summed from 1 to 2. For example, \( \varphi_{,i} \vartheta^i = \sum_{i=1}^3 \frac{\partial \varphi}{\partial x_i} \frac{\partial \vartheta}{\partial x_i} \), and \( \varphi_{,\alpha} A_{\alpha\beta} \vartheta^\beta = \sum_{\alpha=1}^2 \sum_{\beta=1}^2 \frac{\partial \varphi}{\partial x_\alpha} A_{\alpha\beta} \frac{\partial \vartheta}{\partial x_\beta} \).

1.1.2 The Equations

In these notations, the equations of motion are

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0 \quad \text{in } \Omega(t), \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \text{div} S &= f \quad \text{in } \Omega(t).
\end{align*}
\]

Here, \((A \otimes B)_{ij} = A_i B_j\), \((\text{div } M)_i = M_{i,j}\).

By direct computation

\[
\partial_t (\rho u)_i + [\text{div}(\rho u \otimes u)]_i = \partial_t \rho u_i + \rho \partial_t u_i + \rho_j u_i u_j + \rho u_i u_{j,j} + \rho u_i u_{i,j}
= u_i (\partial_t \rho + \rho_j u_j + \rho u_{j,j}) + \rho (\partial_t u_i + u_{i,j} u_j)
= \rho (\partial_t u + u \cdot \nabla u)_i.
\]

It follows that the system (1.1) is equivalent to

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0 \quad \text{in } \Omega(t), \\
\rho \partial_t u + u \cdot \nabla u + \text{div } S &= f \quad \text{in } \Omega(t),
\end{align*}
\]

where \((u \cdot \nabla u)_i = u_j u_{i,j}\).

The interests mainly lie on two types of fluid.

Type 1: Incompressible.

For the incompressible fluid, we assume (i) \( \text{div } u = 0 \) in \( \Omega(t) \), and (ii) the stress tensor is \( S = p I - \mu \nabla u \) where \( I \) is the \( 3 \times 3 \) identity matrix, \( \nabla u = Du + (Du)^T \) is the symmetric gradient, and \( \mu \geq 0 \) is the \textit{(shear)} viscosity.
Under these assumptions, we have that
\[ \text{div}(\rho u) = \nabla \rho \cdot u + \rho \text{div} u = \nabla \rho \cdot u, \]
and that
\[ (\text{div} S)_i = [\text{div}(pI - \mu Du + \mu (Du)^T)]_i = p_{i,j} - \mu u_{i,j,j} - \mu u_{j,i,j} = (\nabla p - \mu \Delta u)_i . \]

It follows that the system (1.2) reduces to
\begin{align*}
\partial_t \rho + \nabla \rho \cdot u &= 0 \quad \text{in } \Omega(t), \\
\rho(\partial_t u + u \cdot \nabla u) + \nabla p - \mu \Delta u &= f \quad \text{in } \Omega(t), \\
\text{div} u &= 0 \quad \text{in } \Omega(t).
\end{align*}

Notice that if we assume \( \rho \) is constant, the first equation holds trivially. In these notes, when incompressibility is taken into account, we always assume this. When \( \mu > 0 \), we call the system viscous (incompressible Navier-Stokes equation). When \( \mu = 0 \), we call the system inviscid (incompressible Euler equation).

**Type 2:** Compressible.

For compressible fluid, we assume (i) \( S = pI - \mu D u - \lambda \text{div} u I \), and (ii) \( p = P(\rho) \), where \( \mu \) is the shear viscosity, \( \lambda \) is the bulk viscosity satisfying \( \lambda + \frac{\mu}{3} \geq 0 \), and \( P: (0, \infty) \to \mathbb{R} \) is a smooth function.

Under these assumptions, we have that
\[ (\text{div} S)_i = [\text{div}(pI - \mu D u - \lambda \text{div} u I)]_i = p_{i,j} - \mu u_{i,j,j} - \mu u_{j,i,j} - \lambda u_{j,i,j} = [\nabla P(\rho) - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u]_i . \]

Then it follows that the system (1.2) reduces to
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0 \quad \text{in } \Omega(t), \\
\rho(\partial_t u + u \cdot \nabla u) + \nabla P(\rho) - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u &= f \quad \text{in } \Omega(t).
\end{align*}

Notice when \( \mu > 0 \) and \( \lambda \geq -\frac{\mu}{3} \), we call the system viscous (compressible Navier-Stokes equation). When \( \mu = \lambda = 0 \), we call the system inviscid (compressible Euler equation).

### 1.1.3 Interpretation of equations

**Claim:** The first equations in (1.3) and (1.4) describe the conservation of mass, and the second equations in (1.3) and (1.4) illustrate the balance of momentum (according to Newton’s law \( F = ma \)).

To see why these are true, we need two tools.

**Tool 1:** The flow map.

The idea of flow maps is to track the motion of particles in the fluid by solving the ODE
\[ \partial_t \xi(x,t) = u(\xi(x,t),t), \]
\[ \xi(x,0) = x. \quad (1.5) \]

We denote the region to which \( E \subset \Omega(0) \) goes at time \( t \) by \( \xi(E,t) \). We need some facts from the theory of ODE. The first fact is that \( \xi(\cdot,t) \) is a orientation preserving diffeomorphism. Another fact we would like to
use is the Liouville’s theorem, which says that the determinant of $D\xi$ is given by
\[
\det D\xi(x, t) = \exp \left( \int_0^t \text{div} u(\xi(x, s), s) \, ds \right),
\]
who is the solution to
\[
\partial_t (\det D\xi(x, t)) = \text{div} u(\xi(x, t), t) \det D\xi(x, t),
\]
\[
\det D\xi(x, t) = 1.
\]

**Tool 2:** Reynold’s transport theorem.

We state the theorem as follows.

**Theorem 1.1** (Reynold). Let $\xi$ be the flow map of $u$ and $U(0) \subset \Omega(0)$. For a (sufficiently regular, in fact, $C^1$ will do) map $\varphi(\cdot, t): \Omega(t) \to \mathbb{R}^m$ ($m \geq 1$), we have
\[
\frac{d}{dt} \int_{U(t)} \varphi(x, t) \, dx = \int_{U(t)} \partial_t \varphi(x, t) + \text{div}(\varphi \otimes u)(x, t) \, dx.
\]

Note: the term $\text{div}(\varphi \otimes u)$ means $[\text{div}(\varphi \otimes u)]_i = (\varphi_i u_j)_j$ for $i = 1, \cdots, m$.

**Proof.** By change of variable $x \mapsto \xi(y, t)$, Liouville’s theorem, and (1.5), we obtain
\[
\frac{d}{dt} \int_{U(t)} \varphi(x, t) \, dx = \frac{d}{dt} \int_{U(0)} \varphi(\xi(y, t), t) \det D\xi(y, t) \, dy
\]
\[
= \frac{d}{dt} \int_{U(0)} \varphi(\xi(y, t), t) \exp \left( \int_0^t \text{div} u(\xi(y, s), s) \, ds \right) \, dy
\]
\[
= \int_{U(t)} \partial_t \varphi(x, t) \, dx
\]
\[
+ \int_{U(0)} \left[ \nabla \varphi(\xi(y, t), t) \cdot \partial_t \xi(y, t) + \varphi(\xi(y, t), t) \cdot \text{div} u(\xi(y, t), t) \right] \det D\xi(y, t) \, dy
\]
\[
= \int_{U(t)} \partial_t \varphi(x, t) \, dx
\]
\[
+ \int_{U(0)} \left[ \nabla \varphi(\xi(y, t), t) \cdot u(\xi(y, t), t) + \varphi(\xi(y, t), t) \cdot \text{div} u(\xi(y, t), t) \right] \det D\xi(y, t) \, dy
\]
\[
= \int_{U(t)} \partial_t \varphi(x, t) + \text{div}(\varphi \otimes u)(x, t) \, dx.
\]

With these tools, we can justify the claims about mass and momentum.

**Mass conservation**

If we take $\varphi = \rho$ where $m = 1$ in Reynold’s theorem, we obtain
\[
\frac{d}{dt} \int_{U(t)} \rho(x, t) \, dx = \int_{U(t)} \partial_t \rho + \text{div}(\rho u) \, dx = 0,
\]
and hence, for any \( U(0) \subset \Omega(0), U(t) = \xi(U(0), t) \), we have that the mass is conserved along the flow

\[
\int_{U(t)} \rho(x, t) \, dx = \int_{U(0)} \rho(x, 0) \, dx.
\]

**Volume conservation** (if assume incompressibility)

When \( \text{div} \ u = 0 \), let \( \varphi \equiv 1 \), then

\[
\frac{d}{dt} \int_{U(t)} \, dx = \int_{U(t)} \text{div} \, u \, dx = 0,
\]

which says that incompressible flows preserve volumes.

**Balance of momentum**

Let \( \varphi = \rho u \) be the momentum density, we have that

\[
\frac{d}{dt} \int_{U(t)} \rho u \, dx = \int_{U(t)} \partial_t (\rho u) + \text{div}(\rho u \otimes u) \, dx = \int_{U(t)} f - \text{div} \, S \, dx.
\]

Notice that the right-hand-side are the forces acting on \( U(t) \). In fact, \( \int_{U(t)} f \, dx \) is the bulk (or exterior) force, and \( \int_{U(t)} -\text{div} \, S \, dx = \int_{\partial U(t)} -S \nu \, d\omega \) is the contact force on \( U(t) \) due to contact with \( \Omega(t) \setminus U(t) \) (\( \nu \) is the outward normal vector). This reveals that the second equation is Newton’s second law in disguise

\[
\frac{d}{dt} \text{Momentum} = \text{Force} \text{ (or sum of)}.
\]

### 1.2 Boundary conditions

#### 1.2.1 Rigid boundary

At the interface between a fluid and a rigid unmoving solid, the boundary conditions are

\[
\begin{align*}
 u \cdot \nu &= 0 & \text{on rigid boundary for inviscid fluid (no penetration),} \\
 u &= 0 & \text{on rigid boundary for viscous fluid (no slip).}
\end{align*}
\]

Notice that the “no-slip” condition is standard assumption, but not entirely from the first principles.

#### 1.2.2 Free boundary

We study the 2D interface between two fluids (see Figure 1.1). The first assumption is that fluids are immiscible, i.e., do not mix. Let \( \nu \) be the unit normal vector pointing into \( \Omega_+ (t) \). Given some continuous functions \( \varphi_{\pm} \) defined in \( \Omega_{\pm} (t) \), we define the jump in \( \varphi \)

\[
\llbracket \varphi \rrbracket = (\varphi_+ - \varphi_-)|_{\Sigma(t)}.
\]

Notice that \( \varphi \) is continuous in \( \Omega_+ (t) \cup \Omega_- (t) \) only if \( \llbracket \varphi \rrbracket = 0 \). The conditions at a free interface \( \Sigma(t) \) are essentially jump conditions

\[
\begin{align*}
 \llbracket u \cdot \nu \rrbracket &= 0 & \text{on } \Sigma(t) \text{ for inviscid problem,} \\
 \llbracket u \rrbracket &= 0 & \text{on } \Sigma(t) \text{ for viscous problem,}
\end{align*}
\]

and

\[
\llbracket S \nu \rrbracket = \sigma H \nu \quad \text{on } \Sigma(t),
\]
\[ \Omega^- (t) \] 

\[ \Omega^+ (t) \] 

\[ \Sigma (t) \]

Figure 1.1: Free interface between two fluids.

where \( S_\pm \) are stress tensors, \( \sigma \geq 0 \) is the surface tensor coefficients, and \( H \) is the mean curvature of \( \Sigma (t) \). In other words, \( \sigma H \nu \) models surface tension. In fact, \(-H\) is the first variation of the area functional, so the equation \( \llbracket 5 \nu \rrbracket = \sigma H \nu \) tells that there is a balance of forces

\[ -S_+ \nu + S_- \nu + \sigma H \nu = 0. \]

We interpret \(-S_+ \nu \) as the force of \( \Omega^- \) on \( \Omega^+ \), \( S_- \nu \) as the force of \( \Omega^+ \) on \( \Omega^- \), and \( \sigma H \nu \) as the force that “punishes” increase in area.

1.3 Boundary Kinematics

In particular, the flow map tells us everything we want about \( \Omega (t) \): \( \Omega (t) = \xi (\Omega (0), t) \). In practice, it’s useful to have other ways of tracking \( \Omega (t) \). For example, if \( \Omega_\pm (t) \) have a common interface, which is the graph of a function, we could use the graph formulation.

We consider two layers of fluid, and assume \( \Sigma (t) = \{ x \in \mathbb{R}^3 \mid x_3 = \eta (x', t) \} \) with \( \eta : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R} \), \( x' = (x_1, x_2) \). Our goal is to find equations for the evolution of \( \eta \).

It’s not hard to find that \( \eta \) satisfies

\[ \xi_3 (y, t) = \eta (\xi'(y, t), t) \quad \text{for} \quad \xi' = (\xi_1, \xi_2). \]

Hence, the kinematic transport equation for \( \eta \) is

\[ \partial_t \eta + u_\alpha |\Sigma (t) \eta_\alpha = u_3 |\Sigma (t), \] (1.6)

In this equation, we observe that

\[ u_3 - u_\alpha \eta_\alpha = u \cdot (-\nabla \eta, 1) = (u \cdot \nu) \sqrt{1 + |\nabla \eta|^2}, \]

and that the boundary conditions guarantee that \( \llbracket u \cdot \nu \rrbracket = 0 \) on \( \Sigma (t) \), which says that \( u \cdot \nu |_{\Sigma (t)} \) is well-defined.

As a consequence, the kinematic transport equation (1.6) is equivalent to

\[ \partial_t \eta = (u \cdot \eta) |_{\Sigma (t)} \sqrt{1 + |\nabla \eta|^2}. \]

The graph formulation has a key benefit: for a graph \( x_3 = \eta (x', t) \), the mean curvature of \( \Sigma (t) \) is

\[ H = H (\eta) = \text{div}_{x'} \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}}. \]

It follows that the area functional

\[ A (\eta) = \int_{\mathbb{R}^2} \sqrt{1 + |\nabla \eta|^2} \; dx'. \]
has first variation

\[ \langle DA(\eta), \varphi \rangle = \lim_{\epsilon \to 0} \frac{d}{d\epsilon} A(\eta + \epsilon \varphi) = \int_{\mathbb{R}^2} \frac{\nabla \eta \cdot \nabla \varphi}{\sqrt{1 + |\nabla \eta|^2}} \, dx' = \int_{\mathbb{R}^2} \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) \varphi \, dx' = \langle -H, \varphi \rangle. \]

This tells us that \( DA(\eta) = -H(\eta) \), which is saying that the surface tension decreases \( L^2 \) area of the surface.

### 1.4 The two layer problem

The rest of the lecture notes will concern the following problem (see Figure 1.2): if we have two layers of distinct, immiscible fluid that are put on top of each other, and sandwiched by a horizontal box, what would happen? To formulate this problem, we denote \( \Omega_+(t) = \{ x \in \mathbb{R}^3 \mid \eta(x', t) < x_3 < l \} \) and \( \Omega_-(t) = \{ x \in \mathbb{R}^3 \mid -l < x_3 < \eta(x', t) \} \), with two rigid boundaries \( \Sigma_+ = \{ x \in \mathbb{R}^3 \mid x_3 = l \} \) and \( \Sigma_- = \{ x \in \mathbb{R}^3 \mid x_3 = -l \} \), and the free surface \( \Sigma(t) = \{ x \in \mathbb{R}^3 \mid x_3 = \eta(x', t) \} \). The top fluid is described by \((\rho_+, u_+, p_+, \mu_+)\) and the bottom fluid is described by \((\rho_-, u_-, p_-, \mu_-)\). The only external force exerted on fluids is the gravity force \( f_\pm = -g \rho_\pm e_3 \) for \( g > 0 \). The fluids might have surface tension with surface tensors \( \sigma_\pm \).

Figure 1.2: The two layer problem.

The equations of motion are

\[
\begin{align*}
\partial_t \rho_\pm + \text{div}(\rho_\pm u_\pm) &= 0 \quad \text{in } \Omega_\pm(t), \quad (1.7a) \\
\rho_\pm (\partial_t u_\pm + u_\pm \cdot \nabla u_\pm) + \text{div} S_\pm &= -g \rho_\pm e_3 \quad \text{in } \Omega_\pm(t). \quad (1.7b)
\end{align*}
\]

Boundary conditions on the free surface are

\[
\begin{align*}
\|u\cdot\nu\| &= 0 \quad \text{on } \Sigma(t) \text{ for inviscid problem,} \\
\|u\| &= 0 \quad \text{on } \Sigma(t) \text{ for viscous problem,}
\end{align*}
\]

and for \( \sigma \geq 0 \)

\[ \|S \nu\| = \| (\rho I - \mu D u) \nu \| = \sigma H(\eta) \nu \quad \text{on } \Sigma(t). \]

In the graph formulation, the kinematic transport equation for \( \eta \) is

\[ \partial_t \eta + u_\alpha \eta,\alpha = u_3 \quad \text{on } \Sigma(t), \]
and boundary conditions on the rigid boundaries are

\[ \mathbf{u}_\pm \cdot \mathbf{e}_3 = 0 \quad \text{on } \Sigma_\pm \text{ for inviscid problem}, \]
\[ \mathbf{u}_\pm = 0 \quad \text{on } \Sigma_\pm \text{ for viscous problem}. \]

From now on, we only consider the two-layer incompressible problem. We scaler product (1.7b) with \( u \) and integrate with respect to \( x \)

\[ \frac{d}{dt} \int_{\Omega(t)} \rho \left\| \mathbf{u} \right\|^2 dx + \int_{\Omega(t)} \rho (u \cdot \nabla u) \cdot u + \nabla \rho \cdot \mathbf{u} dx - \int_{\Omega(t)} \mu \Delta \mathbf{u} \cdot \mathbf{u} dx = -g \int_{\Omega(t)} \rho \mathbf{e}_3 \cdot \mathbf{u} dx, \]

Here, for any functions \( \varphi_\pm \) defined on domains \( \Omega_\pm(t) \), we denote

\[ \int_{\Omega(t)} \varphi = \int_{\Omega_+(t)} \varphi + \int_{\Omega_-(t)} \varphi. \]

Using integration by parts and incompressibility of the fluids, we see that

\[ \int_{\Omega(t)} \rho \left( u \cdot \nabla u \right) \cdot u + \nabla \rho \cdot \mathbf{u} dx - \int_{\Omega(t)} \mu \Delta \mathbf{u} \cdot \mathbf{u} dx = \frac{d}{dt} \int_{\mathbb{R}^2} \sigma \sqrt{1 + \left| \nabla \right|^2} dx' + \int_{\Omega(t)} \frac{\mu}{2} \left| D \mathbf{u} \right|^2 dx. \]

For the last term, by using integration by parts, the boundary conditions and the kinematic transport equation, we obtain

\[ -g \int_{\Omega(t)} \rho \nabla x_3 \cdot \mathbf{u} dx = -g \int_{\Sigma(t)} \rho_+ x_3 u_+ \cdot \frac{(\eta_1, \eta_2, -1)}{\sqrt{1 + \left| \nabla \right|^2}} dx - g \int_{\Sigma(t)} \rho_- x_3 u_- \cdot \frac{(\eta_1, \eta_2, 1)}{\sqrt{1 + \left| \nabla \right|^2}} dx \]
\[ = g \int_{\mathbb{R}^2} \rho_+ \eta_h dx' - g \int_{\mathbb{R}^2} \rho_- \eta_h dx' \]
\[ = \frac{d}{dt} \int_{\mathbb{R}^2} \frac{g \left| \rho \right|}{2} \left| \eta \right|^2 dx'. \]

Therefore, for the two-layer incompressible problem, we find that

\[ \frac{d}{dt} \left( \int_{\Omega(t)} \rho \left\| \mathbf{u} \right\|^2 dx + \int_{\mathbb{R}^2} \left( -\frac{g \left| \rho \right|}{2} \left| \eta \right|^2 + \sigma \sqrt{1 + \left| \nabla \eta \right|^2} \right) dx' \right) + \int_{\Omega(t)} \frac{\mu}{2} \left| D \mathbf{u} \right|^2 dx = 0. \]

Notice that the second integral on the left is the sum of the gravity potential and the area functional, and the third term is the nonnegative dissipation term. This shows that the quantity

\[ E(t) = \int_{\Omega(t)} \rho \left\| \mathbf{u} \right\|^2 dx + \int_{\mathbb{R}^2} \left( -\frac{g \left| \rho \right|}{2} \left| \eta \right|^2 + \sigma \sqrt{1 + \left| \nabla \eta \right|^2} \right) dx' \]

in non-increasing as time evolves.
2 Fundamentals of the two-layer problem

2.1 Flattening

2.1.1 New variables

Instead of dealing with the two-layer problem in time dependent domains, we would like to flatten the free surface by using some new variables (see Figure 2.1). If we let $\Omega_+ = \{x \in \mathbb{R}^3 \mid 0 < x_3 < l\}$ and $\Omega_- = \{x \in \mathbb{R}^3 \mid -l < x_3 < 0\}$ and define the flattening map $\Phi_\pm : \bar{\Omega}_\pm \times [0, \infty) \to \mathbb{R}^3$ by

$$\Phi_\pm(x,t) = (x_1, x_2, x_3 + (1 \mp \frac{x_3}{l})\eta(x', t)),$$

where

$$\Sigma_+ = \{x_3 = l\}$$

$$\Omega_+$$

$$\Sigma = \{x_3 = 0\}$$

$$\Omega_-$$

$$\Sigma_- = \{x_3 = -l\}$$

Figure 2.1: Flattened fluid domains.

We observe that

(i) $\Phi_\pm(x', 0, t) = (x_1, x_2, \eta(x', t))$, and in other words, $\Phi_\pm(\Sigma, t) = \Sigma(t)$;

(ii) $\Phi_\pm(x', \pm l, t) = (x_1, x_2, \pm l)$, and in other words, $\Phi_\pm|_{\Sigma_\pm} = I_{\Sigma_\pm}$ (identity map on $\Sigma_\pm$);

(iii) $\Phi(\cdot, t)$ are invertible maps as long as $\|\eta(\cdot, t)\|_{L^\infty} < l$ (free surface does not touch the boundaries of the box), and in that case

$$\Phi_\pm^{-1}(y, t) = (y', \frac{y_3 - \eta(y', t)}{1 \mp \eta(y', t)/l}).$$

This is also saying that $\Phi_\pm(\cdot, t) : \Omega_\pm \to \Omega_\pm(t)$ is a diffeomorphism.

By direct computation, we find that

$$D\Phi_\pm = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A_\pm & B_\pm & J_\pm \end{pmatrix}, \quad A_\pm = (D\Phi_\pm)^{-T} = \begin{pmatrix} 1 & 0 & -A_\pm K_\pm \\ 0 & 1 & -B_\pm K_\pm \\ 0 & 0 & K_\pm \end{pmatrix},$$

where

$$A_\pm = W_\pm \eta_1,$$

$$B_\pm = W_\pm \eta_2,$$

$$J_\pm = 1 \mp \eta/l,$$

$$K_\pm = 1/J_\pm,$$

$$W_\pm = 1 \mp \frac{x_3}{l}.$$

For some function $f(x, t) = \bar{f}(\Phi(x, t), t)$, if we suppress $\pm$, it’s clear that

$$\begin{cases} \partial_t f - KW\partial_\eta f_{,3} = \partial_t \bar{f} \circ \Phi, \\ A_{ij} f_{,j} = \bar{f}_{,i} \circ \Phi. \end{cases}$$
In this spirit, if we know that $\bar{u}_\pm, \bar{p}_\pm, \bar{\eta}$ solve the two-layer problem on the non-flattened domains, we could define corresponding functions $u_\pm, p_\pm, \eta: \Omega_\pm \times [0, \infty) \rightarrow \mathbb{R}^3 \times \mathbb{R}$ on the flattened domain $u_\pm(x,t) = \bar{u}_\pm(\Phi_\pm(x,t),t)$,
\[ p_\pm(x,t) = \bar{p}_\pm(\Phi_\pm(x,t),t), \]
\[ \eta(x',t) = \bar{\eta}(x',t). \]

2.1.2 New notations

We define some new operators that act on some function $f$, some vector $X$ or $u$, and some 2-tensor $T$
\[(\nabla_A f)_i = A_{ij} f_j,\]
\[\text{div}_A X = A_{ij} X_{i,j},\]
\[(\mathbb{D}_A u)_{ij} = A_{ik} u_{j,k} + A_{jk} u_{i,k},\]
\[(\text{div}_A T)_i = A_{jk} T_{ij,k}.\]

2.1.3 New equations

With the new variables that we defined in §2.1.1, the two-layer problem becomes
\[\rho_\pm (\partial_t u_\pm - K_\pm W_\pm \partial_t \eta(u_\pm),3 + u_\pm \cdot \nabla_{A_\pm} u) + \text{div}_{A_\pm} S_{A_\pm} = -g\rho_\pm e_3 \quad \text{in } \Omega_\pm, \quad (2.1a)\]
\[\text{div}_{A_\pm} u_\pm = 0 \quad \text{in } \Omega_\pm, \quad (2.1b)\]

where
\[S_{A_\pm} = p_\pm I - \mu_\pm \mathbb{D}_{A_\pm} u_\pm.\]

The boundary conditions on the interface are
\[[u \cdot \nu] = 0 \quad \text{on } \Sigma \text{ for inviscid problem,} \quad (2.2a)\]
\[[u] = 0 \quad \text{on } \Sigma \text{ for viscous problem,} \quad (2.2b)\]

and for $\sigma \geq 0$
\[\|[S_{A} \nu]\| = \sigma H \nu \quad \text{on } \Sigma, \quad (2.3)\]

where
\[\nu = \frac{-\nabla \eta, 1}{\sqrt{1 + |\nabla \eta|^2}}, \quad H(\eta) = \text{div} \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right).\]

The kinematic transport equation for $\eta$ is
\[\partial_t \eta + u_\alpha \eta_{,\alpha} = u_3 \quad \text{on } \Sigma. \quad (2.4)\]

As we will see, the advantage of this new formulation is that we could use standard PDE techniques to attach this problem, since the domains are no longer time dependent. As a trade-off, the PDE now becomes quasilinear.

2.2 Equilibrium solutions

To study instability of the waves, we need to look for equilibrium solutions with a flat interface $\eta = 0$. 
2.2.1 Inviscid problem

We assume that $u_\pm = U_\pm e_1$ for $U_\pm \in \mathbb{R}$ being some constants. To check that this solution could be an equilibrium solution, we observe that

\[
\begin{align*}
\text{div}_{A_\pm} u_\pm &= 0 \quad \text{in } \Omega_\pm, \\
[u \cdot \nu] = [u \cdot e_3] &= 0 \quad \text{on } \Sigma, \\
\partial_t \eta + u_\alpha \eta,\alpha &= 0 = u_3 \quad \text{on } \Sigma.
\end{align*}
\]

It remains to check (2.1a) and (2.3), where we need

\[
\nabla_{A_\pm} p_\pm = -g \rho_\pm e_3, \\
[p] = 0.
\]

Notice that since $\eta = 0$, $A_\pm = I_\pm$, and therefore, $p_\pm = -g \rho_\pm x_3$.

We conclude that we have equilibrium solutions for the inviscid two-layer problem

\[
\begin{align*}
u_\pm &= U_\pm e_1 \quad \text{(shear flow)}, \\
\eta &= 0 \quad \text{(flat boundary)}, \\
p_\pm &= -g \rho_\pm x_3 \quad \text{(hydrostatic pressure)}.
\end{align*}
\]

2.2.2 Viscous problem

Similarly, we assume that $\eta = 0$ and $u_\pm = U_\pm e_1$ for $U_\pm \in \mathbb{R}$ being some constants. We easily have that (2.1b) and (2.4) are naturally satisfied. Equations (2.1a) and (2.3) give the same solution for $p_\pm$. However, the “no-slip” boundary condition on $\Sigma$ (2.2b) forces $U_\pm = 0$. Thus, we have equilibrium solutions for the viscous two-layer problem

\[
\begin{align*}
u_\pm &= 0 \quad \text{(non-moving fluid)}, \\
\eta &= 0 \quad \text{(flat boundary)}, \\
p_\pm &= -g \rho_\pm x_3 \quad \text{(hydrostatic pressure)}.
\end{align*}
\]

2.3 Linearization

The problem we would like to study is whether the equilibrium solutions we obtained are stable or unstable.

2.3.1 Motivation

For some smooth function $f : \mathbb{R}^n \to \mathbb{R}$, we consider ODE

\[
\begin{align*}
\dot{x}(t) &= f(x(t)), \\
x(0) &= x_0.
\end{align*}
\]

Suppose that we have equilibrium point $x_0$ satisfying $f(x_0) = 0$, and we set $z(t) = x(t) - x_0$, then

\[
\begin{align*}
\dot{z} &= \dot{x} = f(x) = f(x_0 + (x - x_0)) \\
&= f(x_0 + z) = f(x_0) + Df(x_0)z + [f(x_0 + z) - f(x_0) - Df(x_0)z] \\
&= Df(x_0)z + o(|z|).
\end{align*}
\]
The punchline is that $z = x - x_0$ “almost solves” the linear equation $\dot{z} = Df(x_0)z$, with

$$z(t) = \exp(Df(x_0)t)z(0).$$

The theory in basic dynamical system says that if the spectrum $\text{sp}(Df(x_0)) \subset \{z \in \mathbb{C} | \Re(z) < 0\}$, then $x_0$ is an asymptotically stable node for the nonlinear problem. And if $\text{sp}(Df(x_0)) \cap \{z \in \mathbb{C} | \Re(z) > 0\} \neq \emptyset$, then $x_0$ is nonlinearly unstable.

We would like to play a similar game for PDEs. However, there is no general theory stating that the stability of linear problems controls that of the nonlinear problems. Nevertheless, the stability theory of linear problems is still usually a good indicator of what happens to the nonlinear problems.

### 2.3.2 Linearized equations

We demonstrate an algorithm of linearizing PDEs. Suppose we have an equilibrium point $(u_0, p_0, \eta_0)$ (where we suppress ±), and we go ahead assume that we have a one-parameter family of solutions (see Figure 2.2) to the problem $(-\epsilon_0, \epsilon_0) \ni \epsilon \mapsto (u(\epsilon), p(\epsilon), \eta(\epsilon))$, with $0 \mapsto (u_0, p_0, \eta_0)$. Then we plug this family of solutions into the PDE, differentiate with respect to $\epsilon$, and set $\epsilon = 0$. If we denote

$$\dot{u} = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} u(\epsilon), \quad \dot{p} = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} p(\epsilon), \quad \dot{\eta} = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \eta(\epsilon),$$

then what results from this procedure is the linearized PDE.

![Figure 2.2: Solution manifold.](image)

We illustrate this idea by computing the linearized two-layer problem. We will use following notations

$$u' = \frac{d}{d\epsilon} u(\epsilon), \quad p' = \frac{d}{d\epsilon} p(\epsilon), \quad \eta' = \frac{d}{d\epsilon} \eta(\epsilon).$$

Suppose we have a family of solutions parametrized by $\epsilon$, as is described above, that satisfies (2.1a), then we differentiate the equation with respect to $\epsilon$ and set $\epsilon = 0$. If we denote

$$\rho \left( \partial_t u' - K'W_\partial t \eta u, -KW_\partial t \eta' u, 3 - KW_\partial t \eta u, 3 + u' \cdot \nabla_A u + u \cdot \nabla_A u' + u \cdot \nabla_A u' \right) + \text{div}_A (p I - \mu \mathbb{D}_A u) + \text{div}_A (p' I - \mu \mathbb{D}_A u - \mu \mathbb{D}_A u') = 0. \quad (2.5)$$
Since $\Phi = x + W \eta e_3$, and therefore, $\Phi' = W \eta e_3$. Notice that $A$ satisfies
\[ A^T = (D\Phi)^{-1}, \quad A(\epsilon = 0) = I, \]
it's clear that
\[ (A')^T = -(D\Phi)^{-1} D\Phi' (D\Phi)^{-1}. \]

Then it follows that
\[ \dot{A}^T = -D\dot{\Phi}, \]
which means
\[ \dot{A} = (0 \quad 0 \quad \nabla (W\dot{\eta})). \]

Set $\epsilon = 0$ in (2.5), and notice that $u(\epsilon = 0) = u_0 = U e_1$, $u'(\epsilon = 0) = \dot{u}$, $\eta(\epsilon = 0) = 0$, $\eta'(\epsilon = 0) = \dot{\eta}$, $p(\epsilon = 0) = p_0$, and $p'(\epsilon = 0) = \dot{p}$, then if we put back $\pm$ to the functions, we find the linearized (2.1a)
\[ \rho_\pm (\partial_t \dot{u}_\pm + U_\pm (\dot{u}_\pm),) + \nabla (p_\pm + g \rho_\pm W_\pm \dot{\eta}) - \mu_\pm \Delta \dot{u}_\pm = 0 \quad \text{in } \Omega_\pm. \]

Linearized equations of boundary conditions on $\Sigma$ for (2.2a), (2.2b), and (2.3) are
\[ \begin{align*}
\llbracket \dot{u}_3 - U \dot{\eta}, \rrbracket_1 &= 0 \quad \text{on } \Sigma \text{ for inviscid problem,} \quad \text{(2.6)} \\
\llbracket \dot{u} \rrbracket &= 0 \quad \text{on } \Sigma \text{ for viscous problem,} \quad \text{(2.7)} \\
[pI - \mu D\dot{u}] e_3 &= \sigma \Delta \dot{\eta} e_3 \quad \text{on } \Sigma. \quad \text{(2.8)}
\end{align*} \]

The linearized equations on the rigid boundary are
\[ \begin{align*}
\dot{u}_\pm \cdot e_3 &= 0 \quad \text{on } \Sigma_\pm \text{ for inviscid problem,} \\
\dot{u}_\pm &= 0 \quad \text{on } \Sigma_\pm \text{ for viscous problem.}
\end{align*} \]

Finally, the linearized kinematic transport equation is
\[ \partial_t \dot{\eta} = \dot{u}_3 - U \partial_\tau \dot{\eta} \quad \text{on } \Sigma. \]

Now redefine linearized "pressure" by $\dot{p}_\pm \mapsto \dot{p}_\pm - g \rho_\pm W_\pm \dot{\eta}$, and suppress the dots on $(\dot{u}_\pm, \dot{p}_\pm, \dot{\eta})$, and we find the linearized system
\[ \begin{align*}
\rho_\pm (\partial_t u_\pm + U_\pm (u_\pm),) + \nabla p_\pm - \mu_\pm \Delta u_\pm &= 0 \quad \text{in } \Omega_\pm, \quad \text{(2.9a)} \\
\text{div } u_\pm &= 0 \quad \text{in } \Omega_\pm, \quad \text{(2.9b)} \\
\llbracket u_3 - U \eta, \rrbracket_1 &= 0 \quad \text{on } \Sigma \text{ for inviscid problem,} \quad \text{(2.9c)} \\
\llbracket \dot{u} \rrbracket &= 0 \quad \text{on } \Sigma \text{ for viscous problem,} \quad \text{(2.9d)} \\
[pI - \mu D\dot{u}] e_3 &= ([\rho] g \eta + \sigma \Delta \eta) e_3 \quad \text{on } \Sigma, \quad \text{(2.9e)} \\
\dot{u}_\pm \cdot e_3 &= 0 \quad \text{on } \Sigma_\pm \text{ for inviscid problem,} \quad \text{(2.9f)} \\
u_\pm &= 0 \quad \text{on } \Sigma_\pm \text{ for viscous problem,} \quad \text{(2.9g)} \\
\partial_t \eta &= u_3 - U \eta_1 \quad \text{on } \Sigma. \quad \text{(2.9h)}
\end{align*} \]

3 The inviscid linearized two-layer problem

In this section, we study the instability for the inviscid problem (2.9) with $\mu = 0$ and $U_\pm \in \mathbb{R}$. 

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3.1 Normal mode analysis

A normal mode is a motion where all parts of the system are moving sinusoidally with the same frequency and in phase. And all observed configurations of a system may be generated from its normal modes. Therefore, it suffices to study each normal mode, which has a characteristic frequency, its eigenvalue. This amounts to looking for a “growing mode” \( X(x, t) = X(x)e^{\lambda t} \) and also applying the Fourier transform. We are going to study the normal modes first and then use the Fourier analysis.

We use the normal mode method to analyze the two-layer problem. We first make a normal mode ansatz as follow

\[
\begin{align*}
    u_\pm(x, t) &= v_\pm(\xi, x_3)e^{\lambda t + ix_3}, \\
    p_\pm(x, t) &= q_\pm(\xi, x_3)e^{\lambda t + ix_3}, \\
    \eta(x', t) &= k(\xi)e^{\lambda t + ix_3},
\end{align*}
\]

where \( \lambda \in \mathbb{C}, \xi \in \mathbb{R}^2 \setminus \{0\} \).

We again suppress \( \pm \) for now. Plugging this ansatz into equations (2.9a) and (2.9b) yields

\[
\begin{align*}
    \rho(\lambda + i\xi_1 U)v_1 + i\xi_1 q &= 0, \quad (3.1a) \\
    \rho(\lambda + i\xi_1 U)v_2 + i\xi_2 q &= 0, \quad (3.1b) \\
    \rho(\lambda + i\xi_1 U)v_3 + q_3 &= 0, \quad (3.1c) \\
    i\xi_1 v_\alpha + v_3 &= 0, \quad (3.1d)
\end{align*}
\]

Computing \( i\xi_1 \cdot (3.1a) + i\xi_2 \cdot (3.1b) \) and then using (3.1d) yields

\[
|\xi|^2 q = \rho(\lambda + i\xi_1 U)(i\xi_1 v_1 + i\xi_2 v_2) = -\rho(\lambda + i\xi_1 U)v_3,
\]

which gives

\[
q = -\frac{\rho(\lambda + i\xi_1 U)}{|\xi|^2} v_3.
\]

By plugging it back to (3.1a) and (3.1b), we obtain

\[
v_1 = i\xi_1 v_3, \quad v_2 = i\xi_2 v_3.
\]

Use (3.1c), we then obtain

\[
v_{3,33} = |\xi|^2 v_3,
\]

which is an ODE with solutions (if we put back \( \pm \))

\[
v_3^\pm = A_\pm \cosh (|\xi| x_3) + B_\pm \sinh (|\xi| x_3),
\]

where \( A_\pm \) and \( B_\pm \) will be determined later.

Now consider the jump boundary conditions (2.9c) and (2.9e) on \( \Sigma \), as well as the kinematic transport equation (2.9h). With above computation, we find that these equations on \( \Sigma \) are respectively

\[
\begin{align*}
    \|v_3 - i\xi_1 U\| &= 0, \\
    -\frac{\rho}{|\xi|^2}(\lambda + i\xi_1 U)v_{3,33} &= g[\rho] k - \sigma|\xi|^2 k,
\end{align*}
\]

\[
\lambda k = v_3^+ - i\xi_1 U k = v_3^- - i\xi_1 U k.
\]
These equations on $\Sigma$ are equivalent to
\[
\begin{bmatrix}
v_3 \\
\lambda + i \xi_1 U
\end{bmatrix} = 0,
\tag{3.2a}
\]
\[
\begin{bmatrix}
-\rho(\lambda + i \xi_1 U)v_{3,3} \\
\end{bmatrix} = \frac{g|\xi|^2 \|\rho\| v_3^\pm}{\lambda + i \xi_1 U^\pm} + \frac{\sigma|\xi|^4 v_3^\pm}{\lambda + i \xi_1 U^\pm}.
\tag{3.2b}
\]

Notice that cosh function is even, and therefore, the equation (3.2a) holds if and only if we can write
\[
v_3^\pm = A(\lambda + i \xi_1 U^\pm) \cosh (\xi | x_3) + B^\pm \sinh (\xi | x_3),
\]
fory some $A, B^\pm \in \mathbb{R}$ to be determined.

The boundary conditions (2.9f) on $\Sigma$ are equivalent to
\[
v_3^\pm (\xi, \pm l) = 0.
\tag{3.3}
\]

By equations (3.2b) and (3.3), we find that $A$ and $B^\pm$ must satisfy
\[
\begin{pmatrix}
g|\xi|^2 \|\rho\| - \sigma|\xi|^3 \\
(\lambda + i \xi_1 U^+) \cosh (\xi | l) \\
(\lambda + i \xi_1 U^-) \cosh (\xi | l)
\end{pmatrix}
\begin{pmatrix}
A \\
B^+
\end{pmatrix} = 0.
\]

This linear system having nontrivial solutions if and only if the determinant of the coefficient matrix is zero. Direct computation of the determinant of the matrix shows that the existence of nontrivial solutions to this system is equivalent to
\[
\rho_+(\lambda + i \xi_1 U^+)^2 + \rho_-(\lambda + i \xi_1 U^-)^2 = \tanh (\xi | l) \left( g|\xi|^2 \|\rho\| - \sigma|\xi|^3 \right).
\]

Now write $\lambda = a + ib$ ($a, b \in \mathbb{R}$), then above equation is equivalent to
\[
(a^2 - b^2)(\rho_+ + \rho_-) - b(2\xi_1)(\rho_+ U^+ + \rho_- U^-) - \xi_1^2(\rho_+ U^+_1 + \rho_- U^-_1) = \tanh (\xi | l) \left( g|\xi|^2 \|\rho\| - \sigma|\xi|^3 \right).
\]

Since we are interested in the instability case, we assume that $a \neq 0$. Then we can solve for $b$ from the second equation
\[
b = -\frac{\xi_1(\rho_+ U^+ + \rho_- U^-)}{\rho_+ + \rho_-}.
\]

If we plug this in the first equation, we find that
\[
a^2 = \xi_1^2 \frac{\rho_+ \rho_-}{(\rho_+ + \rho_-)^2 \|u\|^2 + \tanh (\xi | l) \left( g|\xi|^2 \|\rho\| - \sigma|\xi|^3 \right) \rho_+ + \rho_-}.
\]

This equation for $a$ has real solutions with $\Re(\lambda) = a_\pm \neq 0$ (here, $a_\pm$ are two roots of above quadratic equation, it has noting to do with the regions $\Omega_\pm$) if and only if
\[
\xi_1^2 \frac{\rho_+ \rho_-}{(\rho_+ + \rho_-)^2 \|u\|^2 + \tanh (\xi | l) \left( g|\xi|^2 \|\rho\| - \sigma|\xi|^3 \right) \rho_+ + \rho_-} > 0.
\tag{3.4}
\]
3.2 Case of no surface tension $\sigma = 0$

For computational simplicity, we assume that $l = \infty$, and we have $\lim_{l \to \infty} \tanh (|\xi| \ell) = 1$. The picture will not change if $l < \infty$. In fact, $\tanh (|\xi| \ell) < 1$ will be just a constant for each fixed $\xi$, and in some sense, it helps stabilize the wave. We consider in the subsection this inviscid two-layer problem without surface tension. When $\sigma = 0$, (3.4) reduces to

$$\xi_1^2 \frac{\rho_+ \rho_-}{(\rho_+ + \rho_-)^2} \|u\|^2 + \frac{g \|\rho\| |\xi|}{\rho_+ + \rho_1} > 0. \quad (3.5)$$

We will develop our discussion in different cases.

3.2.1 The case $[\rho] > 0$

When $[\rho] > 0$, a heavy fluid is on top of a light fluid. (3.5) is always satisfied, regardless of the values of $U_\pm$, $g$, or $\xi$. This means that for all frequencies $\xi \neq 0$, the equilibrium solutions are unstable. This phenomenon is called the Rayleigh-Taylor instability.

3.2.2 The case $[\rho] < 0$, $[\rho] \neq 0$

When $[\rho] < 0$, a light fluid is on top of a heavy fluid. (3.5) is true if and only if

$$\frac{\xi_1^2}{|\xi|} > - \frac{g [\rho] (\rho_+ + \rho_-)}{\rho_+ \rho_- [u]^2} = - \frac{g [\rho^2]}{\rho_+ \rho_- [u]^2}.$$ 

Notice that under the assumption $[\rho^2] < 0$, the right-hand-side is always positive. This condition could be satisfied by lots of choices of $\xi$'s. This is the Kelvin-Helmholtz instability.

3.2.3 The case $[\rho] < 0$, $[\rho] = 0$

When $[\rho] < 0$ and $[\rho] = 0$, a light fluid lies on top of a heavy fluid and velocity field is continuous across the free surface. We observe that (3.5) will never be satisfied, so we don’t expect linear instability.

3.2.4 The case $[\rho] = 0$

If $[\rho] = 0$ and $[\rho] = 0$, then there is nothing to study, since the problem can be regarded as one homogeneous fluid moving in a tube. The “free surface” is an artificial surface that is not physically existing.

If $[\rho] = 0$ but $[\rho] \neq 0$, from (3.5), the instability occurs whenever $\xi_1 \neq 0$. Again, when it happens, this is the Kelvin-Helmholtz instability.

3.2.5 Rayleigh-Taylor vs. Kelvin-Helmholtz

Notice that when instability happens, $\Re(\lambda) = a$ has two roots

$$a \pm (\xi) = \pm \sqrt{\xi_1^2 \frac{\rho_+ \rho_-}{(\rho_+ + \rho_-)^2} \|u\|^2 + \frac{g [\rho] |\xi|}{\rho_+ + \rho_1}}.$$

Also observe that a growing mode is of the form

$$X(\xi, x_3) e^{a(\xi) + ib \xi} e^{ix \cdot \xi},$$
which means that $a_+ (\xi)$ is the exponential growth rate.

Recall that for pure Rayleigh-Taylor instability occurs (case of $\|\rho\| > 0$ and $\|u\| = 0$)

$$a_+ (\xi) \sim c \sqrt{|\xi|} \quad \text{as } |\xi| \to \infty.$$

While for Kelvin-Helmholtz instability occurs (case of $\|u\| \neq 0$)

$$a_+ (\xi) \sim c |\xi_1| \quad \text{as } |\xi_1| \to \infty. \quad \text{or } \sim c |\xi| \quad \text{as } |\xi_1| \sim |\xi| \to \infty.$$

This illustrates the fact that the Kelvin-Helmholtz instability is presumably stronger than the Rayleigh-Taylor instability.

Another observation is that if $g = 0$, we could still have Kelvin-Helmholtz instability. But we must require $g > 0$ if Rayleigh-Taylor instability happens.

### 3.2.6 Ill-posedness

Assume at least one of $\|\rho\| > 0$ and $\|u\| \neq 0$ is true. The linearized inviscid two-layer problem without surface tension ($\sigma = 0$) is ill-posed in the sense of Hadamard: for any $j \in \mathbb{N}$, $t_0 > 0$, and $\alpha > 0$, there exists data $\{(u^{(n)}, \eta^{(n)}(0))\}_{n=1}^{\infty}$ such that

$$\left\| u^{(n)} (0) \right\|_{H^j} + \left\| \eta^{(n)} (0) \right\|_{H^j} \leq \frac{1}{n},$$

but

$$\left\| u^{(n)} (t) \right\|_{L^2} + \left\| \eta^{(n)} (t) \right\|_{L^2} \geq \alpha \quad \text{for all } t \geq T_0.$$

We omit the proof of this argument, which can be found in [2].

### 3.3 Case with surface tension $\sigma > 0$

Again, we consider the case when $l = \infty$ to simplify computation. The picture also generalize to the case $l < \infty$. Observe that $|\xi_1| \leq |\xi|$, then based on (3.4), a necessary condition for instability is

$$|\xi|^2 \left( \frac{\rho_+ - \rho_-}{(\rho_+ + \rho_-)^2} \right) \|u\|^2 + \frac{g \|\rho\| |\xi| - \sigma |\xi|^3}{\rho_+ + \rho_-} > 0,$$

(3.6)

which is an inequality quadratic in $|\xi|$. Therefore, we need

$$\sigma |\xi|^2 - \frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} \|u\|^2 |\xi| - g \|\rho\| |\xi| - \sigma |\xi|^3 < 0.$$

The inequality (3.6) holds if and only if $|\xi| \in (f_-, f_+)$ (see Figure 3.1), where

$$f_\pm = \frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} \|u\|^2 \pm \sqrt{\Xi} \in \mathbb{R},$$

with discriminant

$$\Xi = \left( \frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} \right)^2 + 4 \sigma g \|\rho\|.$$

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3.3.1 The case $\|\rho\| > 0$

When the heavy fluid lies on top of the light fluid, we again have Rayleigh-Taylor instability. Indeed, if $\|\rho\| > 0$, $\Xi > 0$ for any frequencies $\xi \in B(0, f_+ \subset \mathbb{R}^2$. By choosing frequencies $|\xi_1| = |\xi| \in B(0, f_+)$, we could produce unstable solutions. Comparing to the surface-tension-free case, we see that the surface tension stabilizes high-frequency waves, and the long waves (low-frequency waves) are still unstable.

3.3.2 The case $\|\rho\| = 0$, $\|u\| \neq 0$, or $\|\rho\| < 0$, $\|u\|$ large

If $\|\rho\| = 0$, $\|u\| \neq 0$, or $\|\rho\| < 0$, $\|u\|$ is large enough to guarantee the existence of two roots $f_\pm$, then again we see that the waves with frequencies $\xi \in B(0, f_+ \subset \mathbb{R}^2$ are unstable. This tells us that the Kelvin-Helmholtz instability persists for low frequency waves only.

3.3.3 The case $\|\rho\| < 0$, but $\|u\|$ small

If $\|\rho\| < 0$, but $\|u\|$ is small so that $\Xi < 0$, we do not see Kelvin-Helmholtz instability. This suggests that if the surface tension is large, we expect stability.

4 The viscous linearized two-layer problem

In this section, we study the instability for the viscous problem (2.9) with $\mu > 0$. Notice that with viscosity, the equilibrium solutions require $\|u\| = 0$, which means that Kelvin-Helmholtz instability cannot appear. But as we will see, Rayleigh-Taylor instability is still on the table.

4.1 Normal mode analysis

Notice that when the fluids are viscous, the linearized PDE is

$$\rho_\pm \partial_t u_\pm + \nabla p_\pm = \mu_\pm \Delta u_\pm \quad \text{in } \Omega_\pm.$$

The jump condition becomes

$$\|pI - \mu D u\| e_3 = (g \|\rho\| \eta + \sigma \eta) e_3 \quad \text{on } \Sigma,$$

$$\|u\| = 0 \quad \text{on } \Sigma.$$

And the boundary on the rigid boundary is

$$u_\pm = 0 \quad \text{on } \Sigma_\pm.$$
We use the same normal mode ansatz as before

\[ u_\pm(x, t) = v_\pm(\xi, x_3)e^{\lambda t + ix' \cdot \xi}, \]
\[ p_\pm(x, t) = q_\pm(\xi, x_3)e^{\lambda t + ix' \cdot \xi}, \]
\[ \eta(x', t) = k(\xi)e^{\lambda t + ix' \cdot \xi}, \]

where \( \lambda \in \mathbb{C}, \xi \in \mathbb{R}^2 \setminus \{0\} \).

Plug in the ansatz and we find that

\[ \rho \lambda v_1 + i \xi_1 q = \mu (\partial_{x_3}^2 - |\xi|^2) v_1, \] (4.1a)
\[ \rho \lambda v_2 + i \xi_1 q = \mu (\partial_{x_3}^2 - |\xi|^2) v_2, \] (4.1b)
\[ \rho \lambda v_3 + q, 3 = \mu (\partial_{x_3} - |\xi|^2) v_3, \] (4.1c)
\[ i \xi_\alpha v_\alpha + v_3, 3 = 0. \] (4.1d)

Similarly, suppress \( \pm \), and then \( i \xi_1 \cdot (4.1a) + i \xi_2 \cdot (4.1b) \) with (4.1d) gives

\[ -\rho \lambda v_{3, 3} - |\xi|^2 q = -\mu (\partial_{x_3}^2 - |\xi|^2) v_{3, 3}, \]

and then

\[ q = \left[ \mu (\partial_{x_3}^2 - |\xi|^2) - \rho \lambda \right] v_{3, 3} / |\xi|^2. \]

Plugging back in (4.1a) and (4.1b) yields

\[ v_1 = \frac{i \xi_1}{|\xi|^2} v_{3, 3}, \quad v_2 = \frac{i \xi_2}{|\xi|^2} v_{3, 3}. \]

Then (4.1c) gives an ODE for \( v_3 \)

\[ \left[ \mu (\partial_{x_3}^2 - |\xi|^2) - \rho \lambda \right] \left[ \partial_{x_3}^2 - |\xi|^2 \right] v_3 = 0, \]

which has solutions (where we put back \( \pm \))

\[ v_3^\pm(\xi, x_3) = A_\pm e^{\mp |\xi| x_3} + B_\pm e^{\mp r_\pm x_3} + C_\pm e^{\mp |\xi| x_3} + D_\pm e^{\mp r_\pm x_3}, \]

with \( A_\pm, B_\pm, C_\pm, D_\pm \in \mathbb{R} \) and

\[ r_\pm = \sqrt{|\xi|^2 + \rho \pm / \mu \pm}. \]

Notice that we are concerned with the case \( l = \infty \), and \( v_3^\pm \) must vanish when \( x_3 \to \pm \infty \).

Plug these in the boundary conditions on \( \Sigma \) and \( \Sigma_\pm \), as well as the fact that \( Du_+ = Du_- \) on \( \Sigma \), then we obtain

\[
\begin{pmatrix}
1 & 1 & -1 & -1 \\
|\xi| & r_+ & -2 |\xi|^2 & -2 |\xi|^2 \\
1 & r_+ & \mu_+ (r_+^2 + |\xi|^2) & -\mu_+ (r_+^2 + |\xi|^2) \\
\frac{1}{2} R - C - \alpha_+ & \frac{1}{2} R - r_+ \frac{|\xi|}{|\xi|} & \frac{1}{2} R + C - \alpha_- & \frac{1}{2} R + r_+ \frac{|\xi|}{|\xi|}
\end{pmatrix}
\begin{pmatrix}
A_+ \\
B_+ \\
A_- \\
B_-
\end{pmatrix} = 0,
\]

where
where \( \nu_\pm = \frac{\nu_+}{\rho_+} \) is the kinematic viscosity, and
\[
\alpha_\pm = \frac{\rho_\pm}{\rho_+ + \rho_-} \in [0, 1], \quad \lambda = \frac{g|\xi|}{R^2} \left( (\alpha_+ - \alpha_-) + \frac{|\xi|^2 \sigma}{g(\rho_+ + \rho_-)} \right), \quad C = \frac{|\xi|^2}{\lambda} (\alpha_+ \nu_+ - \alpha_- \nu_-).
\]

For nontrivial solutions existing, we must require
\[
- \left[ \frac{g|\xi|}{\lambda^2} \left( (\alpha_+ - \alpha_-) + \frac{|\xi|^2 \sigma}{g(\rho_+ + \rho_-)} \right) + 1 \right] (\alpha_+ r_+ + \alpha_- r_- - |\xi|)
- 4|\xi| \alpha_+ \alpha_- + 4|\xi|^2 \left( \alpha_+ \nu_+ - \alpha_- \nu_- \right) \left( (\alpha_+ r_+ - \alpha_- r_-) + |\xi| (\alpha_+ - \alpha_-) \right)
+ 4|\xi|^3 \left( \alpha_+ \nu_+ - \alpha_- \nu_- \right)^2 (r_+ - |\xi|) (r_- - |\xi|) = 0.
\]

We now simplify the problem by setting \( \nu_+ = \nu_- = \nu \), then it follows that
\[
r_+ = r_- = r = \sqrt{|\xi|^2 + \frac{\lambda}{\nu}}.
\]

Define
\[
y = r/|\xi|, \quad Q = \frac{g}{|\xi|^3 \nu^2}, \quad \text{and} \quad S = \frac{\sigma}{(\rho_+ + \rho_-)(\nu^4)^{1/3}},
\]
so that we have
\[
Q^{-2/3} S = \frac{|\xi|^2 \sigma}{g(\rho_+ + \rho_-)}.
\]

Thus, we find that nontrivial solutions exist if and only if
\[
y^4 + 4\alpha_+ \alpha_- y^3 + 2(1 - 6\alpha_+ \alpha_-) y^2 - 4(1 - 3\alpha_+ \alpha_-) y + (1 - 4\alpha_+ \alpha_-) + Q(\alpha_- - \alpha_+) + Q^{1/3} S = 0. \tag{4.2}
\]

Whenever the solution exists, we can solve for \( \lambda \)
\[
\lambda = \nu|\xi|^2 (y^2 - 1). \tag{4.3}
\]

Thus, for instability, we will need the roots for (4.2) has the absolute value of real parts greater than 1.

### 4.2 The case \( \nu_+ = \nu_- = \nu, [\rho] = 0 \)

#### 4.2.1 Rayleigh-Taylor instability

When \( \sigma = 0 \), we have \( S = 0 \). And (4.2) reduces to
\[
\varphi(y) = y^4 + 4\alpha_+ \alpha_- y^3 + 2(1 - 6\alpha_+ \alpha_-) y^2 - 4(1 - 3\alpha_+ \alpha_-) y + (1 - 4\alpha_+ \alpha_-) + Q(\alpha_- - \alpha_+) = 0.
\]

We observe that \( y = 1 \) is a root for
\[
\left( (\varphi - Q(\alpha_- - \alpha_+)) \right) (y) = y^4 + 4\alpha_+ \alpha_- y^3 + 2(1 - 6\alpha_+ \alpha_-) y^2 - 4(1 - 3\alpha_+ \alpha_-) y + (1 - 4\alpha_+ \alpha_-) = 0,
\]
and that the slope at \( y = 1 \) is
\[
\left( \left( \varphi - \frac{Q(\alpha_- - \alpha_+) \varphi'}{\varphi} \right) (1) \right)'(1) = \varphi'(1) = 14 - 24\alpha_+ \alpha_- > 0, \quad \forall \alpha_+ + \alpha_- = 1, \alpha_\pm \in [0, 1].
\]

Since \( [\rho] > 0 \) is equivalent to \( \alpha_+ > \alpha_- \), we find that there is only one root \( y(|\xi|) \) is real and \( y(|\xi|) > 1 \). According to (4.3), \( \lambda \) is real and positive and the amplitude of the disturbance will grow exponentially with time. This is saying that the disturbances of all the modes with non-zero wavenumber are growing, which indicates Rayleigh-Taylor instability.
4.2.2 Asymptotic behavior

The asymptotic behavior of \( \lambda(\xi) \) for \(|\xi| \to 0\) and \(|\xi| \to \infty\) corresponds to the limits \( y \to \infty \) and \( y \to 1 \), respectively.

By direct analysis, we find that the asymptotic behaviors

\[
\lambda(\xi) \sim \begin{cases} 
  c_0 |\xi| & |\xi| \sim 0, \\
  c_1 |\xi| & |\xi| \sim \infty,
\end{cases}
\]

where \( c_0, c_1 \) are constants.

Since \( \lambda \) is continuously depending on \(|\xi|\), \( \lambda \) is bounded and has a maximum value obtained at some mode of maximum instability (see Figure 4.1). The mode maximizes \( \lambda \) in amplitude is the most unstable mode.

![Figure 4.1: The dependence of \( \lambda \) on \(|\xi|\).](image)

4.3 The case \( \nu_+ = \nu_-, \sigma > 0, \|\rho\| > 0 \)

When surface tension is taken into account, with more complicated analysis (for details, see [1]), we conclude that instability still occurs, but only for \(|\xi| \in B(0, f_+)\), where

\[ f_+ = \sqrt{\frac{g \|\rho\|}{\sigma}}. \]

The punchline is that, again, surface tension stabilizes short-waves, and the wavenumbers which are stabilized by surface tension are independent of viscosity. Though instability occurs, we still have well-posedness.

4.4 The case \( \nu_+ = \nu_-, \|\rho\| < 0, \sigma = 0 \)

Analysis (in [1]) shows that when \( \alpha_+ < \alpha_- \), the equation (4.2) admits two roots whose real parts are positive, but lie in the interval \((0, 1)\). This suggests that we don’t have instability.

5 Nonlinear analysis

The nonlinear analysis of the stability of two-layer problem is much more complicated. Some results on the Rayleigh-Taylor instability on periodic domains are proved in [7]. In [6], in which the nonlinear sharp stability criteria for the incompressible viscous fluid is discussed. And in [3, 4, 5], results on compressible Navier-Stokes equations are discussed.
References


