FRONTS FOR THE SQG EQUATION: A REVIEW

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ABSTRACT. Temperature discontinuities, or fronts, in the surface quasi-geostrophic (SQG) equations support surface waves. By regularizing the contour dynamics equations, we drive a nonlinear and nonlocal equation, which describes the evolution of SQG fronts. In this survey, we review some recent results on the dynamics of SQG fronts, including a derivation of SQG fronts equation, existence of local and global solutions, and evidence of finite-time singularity formation.

1. Introduction. The 2D surface quasi-geostrophic (SQG) equation is classically written as an active scalar equation

\[ \frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = 0, \]

\[ u = \nabla^\perp (-\Delta)^{-1/2} \theta. \]

Here, \( \theta(x,t) \) with \( x = (x,y) \) is an unknown scalar field, and the velocity field \( u(x,t) \) is determined nonlocally from \( \theta \) by a perpendicular Riesz transform [28]

\[ u(x) = -\frac{1}{2\pi} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^2 \setminus B_\varepsilon(x)} \frac{(x - y)}{|x - y|^3} \theta(y) \, dy. \]

The SQG equation comes from the quasi-geostrophic (QG) equation which describes stratified mid-to-high latitude synoptic scale dynamics in oceanic or atmosphere flows. One of the major hypotheses of flows in this altitude range is that the long-scale dynamics of the fluids is governed by the near balance between the Coriolis force and horizontal pressure gradients [21]. The SQG equation is a reduction of the QG equation when the flows are confined near a surface [15, 20, 24]. Mathematically, the inviscid SQG equation has strong similarity to the 3D Euler equations [5, 6], and the SQG patch problem has a formal resemblance to the vortex patch problem [22]. For analysis of the SQG equation, see [1, 4, 23, 25] and the references cited therein.

By the transport nature of the SQG equation, we may consider a special type of weak solution when \( \theta \) takes on only two distinct constant values \( \theta_+, \theta_- \), so that

\[ \theta(x,t) = \begin{cases} \theta_+ & x \in \Omega^+(t), \\ \theta_- & x \in \Omega^-(t), \end{cases} \]

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for some domain \( \Omega(t) \subset \mathbb{R}^2 \), and thus, to study this type of solutions, we only need to study the evolution of the boundary \( \partial \Omega(t) \). We first define a weak solution of the SQG equation \([25]\).

**Definition 1.1 (Weak solutions of the SQG equation).** A bounded function \( \theta \) is a weak solution of the SQG equation if for any \( \phi \in C_c^\infty(\mathbb{R}^2 \times (0, T)) \), we have

\[
\int_{\mathbb{R}^2 \times [0, T]} \left[ \theta(x, t) \phi_t(x, t) + \theta(x, t) u(x, t) \cdot \nabla \phi(x, t) \right] \, dx \, dt = 0.
\]

When \( \Omega(t) \) is simply connected and \( \partial \Omega(t) \) is a connected regular curve, we distinguish two particular types of domain, which are shown in Figure 1.

1. **Patches**, whose boundary is a smooth, simple, closed curve diffeomorphic to the circle \( \mathbb{T} \), with \( \theta = 0 \) outside the patch.
2. **Half-spaces**, whose boundary is a smooth, simple curve diffeomorphic to \( \mathbb{R} \) that divides \( \mathbb{R}^2 \) into two half-spaces.

In the case of a patch, one can take \( \theta(\cdot, t) = \theta_+ \chi_{\Omega(t)} \) where \( \Omega(t) \) is a simply connected bounded subset of \( \mathbb{R}^2 \), whose boundary is parametrized by \( x = X(\gamma, t) \).

If the constant is normalized by choosing \( \theta_+ = 2\pi \), one obtains well-defined contour dynamics equations for the patch \([11]\)

\[
X_t(\eta, t) = c(\eta, t) \partial_\eta X(\eta, t) + \int_T \frac{\partial_\eta X(\eta, t) - \partial_\eta X(\eta - \zeta, t)}{|X(\eta, t) - X(\eta - \zeta, t)|} \, d\zeta,
\]

where \( c(\eta, t) \) is an arbitrary function corresponding to a time-dependent reparametrization of the curve. Local existence and uniqueness of the initial-value problem of this equation is established in Sobolev spaces by arc-length reparametrization of the patch boundary \([7, 11, 12]\). In \([2, 3, 14]\), a class of nontrivial global solutions are constructed using the Crandall–Rabinowitz’s bifurcation theorem. It has also been proved that splash singularities cannot occur in a smooth boundary of an SQG patch \([13]\), but whether other types of finite-time singularities can occur remains open. Some numerical studies are carried out in \([8, 27]\), where a curvature blow-up on the SQG patch boundary and a complicated self-similar cascade of filament instability are observed in the numerical simulations.

In the case of half-spaces, we refer the boundary \( \partial \Omega(t) \) as a front. In contrast to the patches, the formal contour dynamics equation, obtained by replacing the integration limit \( T \) with \( \mathbb{R} \) in \((1)\), does not converge. However, a regularization
procedure is introduced in [16] to make sense of the divergent integral. We remark that in [19], we justify this procedure by using direct contour dynamics methods. The derivation of a regularized evolutionary equation describing the dynamics of SQG fronts is reviewed in Section 2. When the fronts are spatially periodic, the initial-value problem is proved to be locally well-posed in the $C^\infty$-class, by a Nash–Moser argument [26], and in the analytic class, by a Cauchy–Kowalevski theorem [10]. In [17], the initial value problem for a cubically nonlinear, approximate equation of the regularized equation (2) with periodic initial data is proved to be locally well-posed in Sobolev spaces. We also prove the global well-posedness of the initial value problem for the full equation (5) with a smallness assumption on the initial data [18]. The proof for global well-posedness is outlined in Section 3. Finally, we survey in Section 4 a numerical study of the SQG fronts, which suggests wave breaking or singularity formation in finite time.

2. Regularized SQG front equations. We will consider only fronts that are a graph, located at

$$y = \varphi(x, t),$$

where $\varphi(x, t) : \mathbb{R} \to \mathbb{R}$ is a smooth, bounded function. As is discussed above, the formal contour dynamics equation for the fronts does not converge. To make sense of the equation, we propose the following regularization. We first cut-off the integration region using a ball with radius $\lambda$ around a point $x$ on the front, and then obtain the truncated equation for $X$

$$X_\lambda(\eta, t) = c(\eta, t)\partial_\eta X(\eta, t) + \int_{\eta-\lambda}^{\eta+\lambda} \frac{\partial_\zeta X(\eta, t) - \partial_\zeta X(\zeta, t)}{|X(\eta, t) - X(\zeta, t)|} \, d\zeta.$$

When the front curve is given by a graph $\varphi(x, t)$, the function $c$ can be uniquely solved and we obtain an equation for $\varphi$

$$\varphi_t(x, t) + \int_{-\lambda}^{\lambda} \frac{\varphi_x(x + \zeta, t) - \varphi_x(x, t)}{\zeta} \, d\zeta + \int_{-\lambda}^{\lambda} \left[ \frac{\varphi_x(x + \zeta, t) - \varphi_x(x, t)}{\sqrt{\zeta^2 + (\varphi(x + \zeta, t) - \varphi(x, t))^2}} \right] \frac{\varphi_x(x + \zeta, t) - \varphi_x(x, t)}{|\zeta|} \, d\zeta = 0.$$

Using the Fourier transform, we find that

$$\int_{-\lambda}^{\lambda} \frac{\varphi_x(x + \zeta, t) - \varphi_x(x, t)}{|\zeta|} \, d\zeta = [d(\lambda) - 2(\log \lambda)] \varphi_x(x, t) - 2 \log |\partial_x| \varphi_x(x, t),$$

where $\log |\partial_x|$ is the Fourier multiplier operator with symbol $\log |\zeta|$ and $d(\lambda) \to -2\gamma$ as $\lambda \to \infty$ with $\gamma$ being the Euler–Mascheroni constant. Now, by a $\lambda$-dependent Galilean transformation $x \mapsto x + [d(\lambda) - 2 \log \lambda] t$, the divergent advection term can be removed. In the limit $\lambda \to \infty$, we get a regularized equation for SQG fronts

$$\varphi_t(x, t) - 2 \log |\partial_x| \varphi_x(x, t)$$

$$= - \int_{\mathbb{R}} [\varphi_x(x, t) - \varphi_x(x + \zeta, t)] \left\{ \frac{1}{|\zeta|} - \frac{1}{\sqrt{\zeta^2 + (\varphi(x, t) - \varphi(x + \zeta, t))^2}} \right\} \, d\zeta. \quad (2)$$

Dimensional analysis of the SQG equation demonstrates that parameters $\theta_\pm$ are velocities, so one might expect that the waves on an SQG front are nondispersive.
Nevertheless, this equation is dispersive with dispersion relation \( \omega(\xi) = 2\xi \log |\xi| \) by looking at the linearized equation (originally pointed out in [26])

\[
\varphi_t(x, t) = 2 \log |\partial_x| \varphi_x(x, t).
\]  

This linearized equation (3) has an anomalous scaling-Galilean invariance \( x \mapsto \lambda[x + 2(\log \lambda)t], \ t \mapsto \lambda t \), and the linearized equation commutes with a scaling-Galilean vector field

\[
\mathcal{S} = (x + 2t)\partial_x + t\partial_t.
\]

3. Global well-posedness of SQG front equation. We consider the initial value problem posed on \( \mathbb{R} \)

\[
\varphi_t(x, t) - 2 \log |\partial_x| \varphi_x(x, t)
= - \int_{\mathbb{R}} \left[ \varphi_x(x, t) - \varphi_x(x + \zeta, t) \right] \left\{ \frac{1}{|\zeta|} - \frac{1}{\sqrt{\zeta^2 + (\varphi(x, t) - \varphi(x + \zeta, t))^2}} \right\} \, d\zeta,
\]

where \( \varphi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \) is defined for \( x \in \mathbb{R} \) and \( t \in \mathbb{R}_+ \). For fronts with small slopes \(|\varphi_x| \ll 1\), we can expand the nonlinearity and rewrite (2) as

\[
\varphi_t(x, t) - 2 \log |\partial_x| \varphi_x(x, t)
= \sum_{n=1}^{\infty} \frac{c_n}{2n+1} \partial_x \int_{\mathbb{R}^{2n+1}} T_n(\eta_1, \ldots, \eta_{2n+1}) \prod_{j=1}^{2n+1} \left( e^{in_j \varphi}(\eta_j, t) \right) \, d\eta_1 \cdots d\eta_{2n+1},
\]

where

\[
T_n(\eta_1, \ldots, \eta_{2n+1}) = \int_{\mathbb{R}} \frac{\prod_{j=1}^{2n+1} (1 - e^{in_j \zeta})}{|\zeta|^{2n+1}} \, d\zeta, \quad c_n = \frac{\sqrt{\pi}}{\Gamma \left( \frac{1}{2} - n \right) \Gamma(n+1)}.
\]

The main theorem in [18] is as follows.

**Theorem 3.1.** Let \( s = 1200, \ r = 7, \) and \( p_0 = 10^{-4} \). There exists a constant \( 0 < \varepsilon \ll 1 \), such that if \( \varphi_0 \in H^s(\mathbb{R}) \) satisfies

\[
\|\varphi_0\|_{H^s} + \|x\varphi_0\|_{H^r} \leq \varepsilon_0
\]

for some \( 0 < \varepsilon_0 \leq \varepsilon \), then there exists a unique global solution \( \varphi \in C([0, \infty); H^s(\mathbb{R})) \) of (5). Moreover, this solution satisfies

\[
\|\varphi(t)\|_{H^s} + \|\partial_x \varphi(t)\|_{H^r} \lesssim \varepsilon_0(t + 1)^{p_0},
\]

where \( \mathcal{S} \) is the vector field in (4).

We remark that according to this theorem, the Sobolev \( H^s \)-norm of \( \varphi \) is not controlled uniformly in time, and it possibly has a very slow growth of order \( (t+1)^{p_0} \). The only norm we can bound uniformly in time is the \( Z \)-norm of \( \varphi \) (see (10) for the definition of \( Z \)-norm). It is not clear to us whether this is a limitation of our method of proof or an intrinsic feature of the solutions.

Following [9], to prove this global existence theorem, it suffices to prove local well-posedness and a suitable global \textit{a priori} bound.
3.1. Local well-posedness. The difficulty in local well-posedness is that straightforward $H^s$-estimates for (5) do not close, due to a logarithmic loss of derivatives [16]. In order to overcome this difficulty, we use Weyl para-differential calculus to para-linearize the equation, then extract a term from the nonlinearity, which can be controlled by the linear term, and define a weighted energy whose estimates do close.

We use $T_{a,f}$ to denote the standard Weyl para-product [29], and define an $s$-order weighted energy as

$$
\dot{E}^{(s)}(t) = \|\varphi\|_{L^2}^2 + \sum_{j=1}^{s} E^{(j)}(t),
$$

$$
E^{(j)}(t) = \int_{\mathbb{R}} |D|^{j} \varphi(x,t) \cdot \left(2 - T_{B^{s_{\varphi}}[\varphi]} \right)^{2j+1} |D|^{j} \varphi(x,t) \, dx,
$$

where, if we denote by $\delta$ the Dirac-delta distribution,

$$
B^{s_{\varphi}}[\varphi](\cdot, \xi) = -F_{\xi}^{-1} \left[ \sum_{n=1}^{\infty} 2c_n \int_{\mathbb{R}^n} \delta \left( \xi - \sum_{j=1}^{2n} \eta_j \right) \cdot \prod_{j=1}^{2n} \left( i\eta_j \varphi(\eta_j) \chi \left( \frac{(2n+1)\eta_j}{\xi} \right) \right) d\eta_1 d\eta_2 \cdots d\eta_{2n} \right].
$$

By carrying out standard estimates for $E^{(s)}(t)$, we find that

$$
E^{(s)}(t) \leq E^{(s)}(0) \exp \left\{ \int_0^t \left[ \left( \|\varphi_x(\tau)\|_{W^{2s,\infty}} + \|\log |\partial_x| \varphi_x(\tau)\|_{W^{2s,\infty}} \right)^2 \right.ight.

\left. \cdot \left. F \left( \|\varphi_x(\tau)\|_{W^{2s,\infty}} + \|\log |\partial_x| \varphi_x(\tau)\|_{W^{2s,\infty}} \right) \right] d\tau \right\},
$$

where, for a sufficiently large number $\bar{C}$ depending only on $s$, $F(\cdot)$ is an increasing continuous real-valued function as long as

$$
\sum_{n=1}^{\infty} \bar{C}^n |c_n| \left( \|\varphi_x(t)\|_{W^{2s,\infty}} + \|\log |\partial_x| \varphi_x(t)\|_{W^{2s,\infty}} \right)^2 < \infty.
$$

The local well-posedness and breakdown criterion for solutions is stated in the following theorem.

**Theorem 3.2.** Let $s > 4$ be an integer. There exists a constant $\bar{C} > 0$, depending only on $s$, such that if $\varphi_0 \in H^s(\mathbb{R})$ satisfies

$$
\|T_{B^{s_{\varphi}}[\varphi_0]}\|_{L^2 \rightarrow L^2} \leq C, \quad \sum_{n=1}^{\infty} \bar{C}^n |c_n| \left( \|\partial_x \varphi_0\|_{W^{2s,\infty}} + \|\partial_x \log |\partial_x| \varphi_0\|_{W^{2s,\infty}} \right) < \infty \quad (9)
$$

for some constant $0 < C < 2$, then there exists a maximal time of existence $0 < T_{\text{max}} \leq \infty$ depending only on $\|\varphi_0\|_{H^s}$, $C$, and $\bar{C}$ such that the initial value problem (5) has a unique solution with $\varphi \in C((0, T_{\text{max}}); H^s(\mathbb{R}))$. If $T_{\text{max}} < \infty$, then either

$$
\lim_{t \rightarrow T_{\text{max}}} \sum_{n=1}^{\infty} \bar{C}^n |c_n| \left( \|\varphi_x(t)\|_{W^{2s,\infty}} + \|\log |\partial_x| \varphi_x(t)\|_{W^{2s,\infty}} \right)^2 = \infty
$$

or

$$
\lim_{t \rightarrow T_{\text{max}}} \|T_{B^{s_{\varphi}}[\varphi(\cdot,t)]}\|_{L^2 \rightarrow L^2} = 2.
$$
We remark that this proof requires the smallness of the $W^{2,\infty}$-norms of $\partial_x \varphi_0$ and $\partial_x \log |\partial_x| \varphi_0$ (see (9)), in order to validate the expansion of nonlinearity, as well as the non-degeneracy of the weight (so that $E^{(s)}(t) \approx \|\varphi\|_{H^s(R)}^2$). It is unclear whether the problem is still locally well-posed if $2 - T_{H^{0,\infty}[\varphi]}$ degenerates.

3.2. Global a priori bound. To complete the proof of global well-posedness theorem, we introduce the $Z$-norm of a function $f$

$$\|f\|_Z = \left\| (|\xi| + |\xi|^{r+3}) \hat{f}(\xi) \right\|_{L_{\xi}^\infty}. \quad (10)$$

To obtain the global a priori bound, we use a bootstrap argument and prove the following proposition.

**Proposition 3.3** (Bootstrap). Let $T > 1$ and suppose that $\varphi \in C([0,T];H^s)$ is a solution of (5), where the initial data satisfies

$$\|\varphi_0\|_{H^s} + \|x\partial_x \varphi_0\|_{H^s} \leq \varepsilon_0$$

for some $0 < \varepsilon_0 \ll 1$. If there exists $\varepsilon_0 \ll \varepsilon_1 \ll \varepsilon_0^{1/3}$ such that the solution satisfies

$$(t + 1)^{-p_0} (\|\varphi(t)\|_{H^s} + \|\mathcal{S}\varphi(t)\|_{H^s}) + \|\varphi\|_Z \leq \varepsilon_1$$

for every $t \in [0,T]$, then the solution satisfies an improved bound

$$(t + 1)^{-p_0} (\|\varphi(t)\|_{H^s} + \|\mathcal{S}\varphi(t)\|_{H^s}) + \|\varphi\|_Z \leq \varepsilon_0.$$  

We call the assumptions in Proposition 3.3 the bootstrap assumptions. To prove Proposition 3.3, the first step is to prove a sharp dispersive estimates

$$\|\varphi(t)\|_{L^\infty} + \|\partial^{r+1} \log |\partial_x| \varphi(t)\|_{L^\infty} \lesssim \varepsilon_1 (t + 1)^{-1/2}.$$  

(11)

We achieve this by first carrying out the standard dispersive estimates for the linearized equation (3) and then taking advantage of the bootstrap assumptions to sharpen it. Using this sharp dispersive estimates, one can directly complete estimates for $\|\varphi(t)\|_{H^s}$. By modifying the weighted energy (8) for $\mathcal{S}\varphi$, we then obtain the improved bounds for

$$(t + 1)^{-p_0} (\|\varphi(t)\|_{H^s} + \|\mathcal{S}\varphi(t)\|_{H^s}).$$

The most difficult part is the nonlinear dispersive estimate, which deals with the estimates for $\|\varphi\|_Z$.

The rest of the proof involves a detailed frequency-space analysis. To show $\|\varphi\|_Z$ is uniformly bounded by a constant of order $\varepsilon_0$, we only need to show that

$$\int_0^T \|\varphi_t\|_Z \, dt = \int_0^T \|h_t\|_Z \, dt \lesssim \varepsilon_0,$$

(12)

where

$$h(x,t) = e^{-2it\partial_x \log |\partial_x| \varphi(x,t)}, \quad \hat{h}(\xi,t) = e^{-2it\xi \log |\xi|} \hat{\varphi}(\xi,t).$$

To this end, we take the Fourier transform of the expanded equation (6), and rewrite the equation as

$$\hat{h}_t + e^{-2it\xi \log |\xi|} \mathcal{N}_{\geq 5}(\varphi)$$

$$= -\frac{1}{6} \int_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{it\Phi(\xi, \eta_1, \eta_2)} \hat{h}(\xi - \eta_1 - \eta_2) \hat{h}(\eta_1) \hat{h}(\eta_2) \, d\eta_1 \, d\eta_2,$$

where $\mathcal{N}_{\geq 5}(\varphi)$ denotes the nonlinearity of quintic degree and higher,

$$\Phi(\xi, \eta_1, \eta_2) = 2(\xi - \eta_1 - \eta_2) \log |\xi - \eta_1 - \eta_2| + 2\eta_1 \log |\eta_1| + 2\eta_2 \log |\eta_2| - 2\xi \log |\xi|.$$
and $\mathbf{T}_1$ is defined as in (7) which can be rewritten as

$$\mathbf{T}_1(\eta_1, \eta_2, \eta_3) = \eta_1^3 \log |\eta_1| + \eta_2^3 \log |\eta_2| + \eta_3^3 \log |\eta_3| + (\eta_1 + \eta_2 + \eta_3)^2 \log |\eta_1 + \eta_2 + \eta_3|$$

$$- (\eta_1 + \eta_2)^2 \log |\eta_1 + \eta_2| - (\eta_1 + \eta_3)^2 \log |\eta_1 + \eta_3| - (\eta_2 + \eta_3)^2 \log |\eta_2 + \eta_3|.$$ 

Therefore, proving inequality (12) is reduced to showing that under the bootstrap assumptions, the integrals

$$\int_0^T \xi(|\xi| + |\xi|^{r+3}) \int_{\mathbb{R}^2} \mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)$$

$$\cdot e^{it\Phi(\xi, -\eta_2)} \hat{h}(\xi - \eta_1 - \eta_2) \hat{h}(\eta_1) \hat{h}(\eta_2) \, d\eta_1 \, d\eta_2 \, dt,$$

(13)

$$\int_0^T (|\xi| + |\xi|^{r+3}) e^{-2d\log |\xi|}/N_{\geq 5}(t) \, dt$$

(14)

are bounded uniformly with respect to $\xi$ and $T$.

For the estimates of (13), we distinguish between time resonances and space resonances. In most regions of frequency space, these resonances do not appear simultaneously, and we can use oscillatory integral estimates or an interpolation inequality to get sufficient decay. To be more specific, in the region with only space resonance, we integrate by parts in the time variable to obtain a fourth degree non-linearity in $h$, and then we use multilinear estimates and sharp dispersive estimates (11) to gain time decay. In the region away from the space-time resonances, we integrate by parts in a frequency variable and gain an extra $(t + 1)^{-1}$ decay. Around the space-time resonances, we need to use modified scattering (a phase correction) to cancel out the leading order non-integrability in time.

The estimates for higher-order nonlinear terms (14) can be proved by multilinear estimates, as the sharp dispersive estimates (11) provides enough time decay.

4. Evidence of singularity formation. We study numerically an approximate model equation of the regularized SQG front equation

$$\varphi_t + \frac{1}{2} \partial_x \left\{ \varphi^2 \log |\partial_x| \varphi_{xx} - \varphi \log |\partial_x| (\varphi^2)_{xx} + \frac{1}{3} \log |\partial_x| (\varphi^3)_x \right\} = 2 \log |\partial_x| \varphi_x.$$  (15)

This equation is obtained by a formal truncation of the nonlinearity of (6) at the cubic level. It is easy to verify that this equation has following two conserved quantities for smooth data

Entropy: $S(t) = \int_T \varphi^2 \, dx$,

Energy: $H(t) = \int_T \varphi \log |\partial_x| \varphi + \frac{1}{8} \varphi^2 \partial_x^2 \log |\partial_x| \varphi^2 - \frac{1}{6} \varphi \partial_x^2 \log |\partial_x| \varphi^3 \, dx$.

We choose initial data

$$\varphi_0(x) = \cos(x + \pi) + \frac{1}{2} \cos[2(x + \pi + 2\pi^2)],$$  (16)

and use a pseudo-spectral method $(2^{15}$ Fourier modes) with spectral viscosity to carry out the numerical simulations. We observe an oscillatory singularity at $t \approx 0.06$ near $x \approx 2.15$ (see Figure 2 and Figure 3).

We remark that a proof of singularity formation for the SQG equation is open.
Figure 2. Left: Solution of (15) with initial data (16), shown at $t = 0$ (blue), $t = 0.01875$ (cyan), $t = 0.0375$ (magenta), $t = 0.05625$ (green), $t = 0.075$ (red). Right: Detail of singularity formation.

Figure 3. Left: Energy of the solution. Right: Entropy of the solutions. Both of these quantities are no longer conserved after time $t \approx 0.06$.

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