DO NOT TURN THIS PAGE UNTIL YOU ARE INSTRUCTED TO BEGIN THE EXAM.

Please sign your name in the following box and read instructions before exam. Unsigned papers will not be graded.

I agree to adhere to the UCD Code of Academic Conduct.

Signature: KEY

Instructions:
This exam contains 9 pages (including this cover page and scratch paper) and 7 problems. Total of points is 100.
The exam will be closed book, closed notes, no calculators and no electronic devices.
You will be graded mainly on whether the answer provided is correct, but also on the work you show to indicate how you arrive at the answer (unless otherwise indicated). If you decide to change your answer, erase or cross out your old answer and neatly write your new answer. Simplify your answers as much as you can. Please BOX your answers.
Tear the last page down for scratch. Raise your hand if you need more scratch paper.
Stop working immediately at the end of the exam when time is called.

Good luck!

Grade Table (for instructor use only)

N/A
N/A
N/A
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1. (10 points) True or False? Circle one, NO WORK NEEDS TO BE SHOWN.

(a) T F Equation $y^{(3)} + t^3 y = e^y$ is an ODE.

(b) T F Function $y = 2 \cos(x)$ is a solution to the differential equation $\frac{d^2 y}{dx^2} + y = 0$.

(c) F T Equation $(y^{(2)} + 5y' + (\sin t)^3(y - 1) = e^{t^2}$ is a 2$^\text{nd}$ order nonlinear ODE.

(d) F T Equation $y^{(2)}y^3 - 5ty' + 1 = 0$ is a 3$^\text{rd}$ order nonlinear ODE.

(e) F T Equation $y' - \sin y + y^2 = 0$ is an autonomous equation.

2. (20 points) Solve following equations.

(a) 
\[
\begin{cases}
\frac{dy}{dt} + \frac{y}{t} = te^t \\
y(1) = e
\end{cases}
\]

Solutions.

It’s a first order linear equation. Integrating factor is $\mu(t) = \exp\left(\int \frac{1}{t} \, dt\right) = t$.

Then we multiply $\mu$ to the equation

$ty' + y = t^2e^t$.

Then

$ty = \int t^2e^t \, dt + C$

$= t^2e^t - 2 \int te^t \, dt + C$

$= t^2e^t - 2te^t + 2e^t + C$.

Plugging in initial condition $y(1) = e$, we obtain $C = 0$. Then the unique solution to the equation is

$y(t) = \frac{1}{t} \left(t^2e^t - 2te^t + 2e^t\right)$. 
(b) \[ y' + e^t y - y^2 = 0. \]

[Hint: \( y = 0 \) is a trivial solution. When \( y \neq 0 \), consider \( u(t) = y^{-1}(t) \). You can express your answer with an indefinite integral.]

**Solutions.**

This is a Bernoulli equation. \( y = 0 \) is a solution. When \( y \neq 0 \), divide the equation by \( y^2 \), we obtain

\[ y^{-2}y' + e^t y^{-1} = 1. \]

Let \( u = y^{-1} \), then \( u' = -y^{-2}y' \). Then the equation is

\[ u' - e^t u = -1. \]

Use integrating factor \( \mu(t) = e^{-\int e^t dt} = e^{-e^t} \), one obtains

\[ \left( e^{-e^t} u \right)' = -e^{-e^t}. \]

Then

\[ u(t) = e^{e^t} \left( -\int e^{-e^t} dt + C \right). \]

Plugging back \( u = y^{-1} \) we have general solution

\[ y(t) = \frac{e^{-e^t}}{C - \int e^{-e^t} dt}, \]

and a particular solution \( y = 0 \).

(c) \[ y' = e^t - y. \]

**Solutions.**

It’s a separable equation. Rewrite this equation into differential form

\[ e^y dy = e^t dt. \]

Integrate both sides

\[ e^y = e^t + C. \]

Then the general solution is

\[ y = \ln \left( e^t + C \right). \]
(d) \[ y' = \frac{t^2 + ty + y^2}{t^2}. \]

[Hint: consider \( u(t) = \frac{y(t)}{t}. \)]

Solutions.
This equation is equivalent to
\[ y' = 1 + \frac{y}{t} + \left( \frac{y}{t} \right)^2. \]

Let \( u = \frac{y}{t} \) and notice that \( y' = u + tu' \), above equation becomes
\[ tu' = 1 + u^2. \]

Notice that this equation is separable, solving this equation gives
\[ \arctan u = \ln|t| + C. \]

Then the general solution is
\[ u = \tan(\ln|t| + C). \]

Substituting back \( u = \frac{y}{t} \) yields
\[ y = t \tan(\ln|t| + C). \]

3. (10 points) Consider the initial value problem \( y' = -\frac{4t}{y} \) with initial data \( y(0) = y_0 \).
Solve this problem and determine how the interval in which the solution exists depends on the initial value \( y_0 \).
Solutions.
If \( y_0 = 0 \), then \( y' = -\frac{4t}{y} \) doesn’t exist. If \( y_0 \neq 0 \), notice equation is separable, then
\[ yy' = -4t. \]

Solve this equation, we obtain
\[ y = \pm \sqrt{-4t^2 + 2C}. \]

Plugging in initial value \( y(0) = y_0 \) we have \( C = y_0^2/2 \). Then
\[ y = \pm \sqrt{-4t^2 + y_0^2}. \]

For \( y_0^2 > 4t^2 \), one obtain If \( y_0 > 0 \), the interval of existence of solution is \((-y_0/2, y_0/2)\).
If \( y_0 < 0 \), the interval of existence of solution is \((y_0/2, -y_0/2)\).
4. (20 points) Solve following equations.

(a) 
\[(\cos x + \frac{1}{y}) + (\frac{1}{y} - \frac{x}{y^2}) \frac{dy}{dx} = 0.\]

*Solutions.*
Let \(M = \cos(x) + 1/y\) and \(N = 1/y - x/y^2\). Then
\[M_y = -\frac{1}{y^2} = N_x.\]
which implies that the equation is exact. Then we want to find a stream function \(\Psi\) such that \(\Psi_x = M\) and \(\Psi_y = N\). Integrating gives
\[\Psi = \sin x + \frac{x}{y} + h(y).\]
Differentiating gives
\[\Psi_y = -\frac{x}{y^2} + h'(y).\]
Therefore, we need \(h'(y) = 1/y\), which gives \(h(y) = \ln|y| + C_1\). Then the general solution is
\[
\sin x + \frac{x}{y} + \ln|y| = C.
\]

(b) 
\[(x + 2) \sin y + (x \cos y) y' = 0.\]

*Solutions.*
Let \(M = (x + 2) \sin y\) and \(N = x \cos y\). We observe that
\[M_y = (x + 2) \cos y, \quad N_x = \cos y.\]
Since \(M_y \neq N_x\), we see that the equation is not exact. Notice that
\[\frac{M_y - N_x}{N} = \frac{1 + x}{x} = 1 + \frac{1}{x}\]
only depends on \(x\). We have that integrating factor \(\mu(x)\) solves
\[\frac{d\mu}{\mu} = 1 + \frac{1}{x} \, dx.\]
Then \(\mu(x) = \exp(x + \ln|x|) = xe^x\). Multiplying \(\mu(x)\), forming an exact derivative, and solving an exact equation gives the general solution
\[x^2e^x \cos y = C.\]
5. (20 points) Consider equation

\[ \frac{dy}{dt} = f(y) := y(y - 1)(y - 3). \]

Sketch the graph of \( f(y) \) versus \( y \), determine the critical (equilibrium) points, and classify each one asymptotically stable or unstable. Draw the phase line, and sketch several graphs of solutions in the \( ty \)-plane.

Solutions.

Solving \( f(y) = 0 \) gives critical points \( y = 0, y = 1, y = 3 \).

Graph of \( f(y) \) versus \( y \) (where \( y \)-axis is the phase line) is

![Graph of f(y) versus y](image)

Stable equilibrium point: 1. Unstable equilibrium point: 0, 3.

Several graphs of solutions:
6. (10 points) Use the forward Euler method with 2 intermediate steps (i.e. \( n = 2 \)) to approximate the solution of

\[
y'(t) = (y(t))^2, y(0) = 1,
\]

up to time \( T = 1 \). Try to solve the initial value problem and compare the estimate with real solution. Explain what happens.

Solutions.

Subdividing interval we get \( t_0 = 0, t_1 = 0.5, \) and \( t_2 = 1. \) Then by forward Euler’s method

\[
y(t_1) - y(t_0) \approx (t_1 - t_0)y_0^2 = 0.5,
\]

then \( y(t_1) \approx 1.5, \) and

\[
y(t_2) - y(t_1) \approx (t_2 - t_1)(y(t_1))^2 = 9/8,
\]

then \( y(1) \approx \frac{21}{8}. \)

To solve this equation, notice that the equation is separable. We can obtain the unique solution

\[
y(t) = \frac{1}{1 - t}.
\]

We see that \( y(1) \) does not exist.

Explanation: Even if the solution does not exist at \( t = 1, \) we can still find an approximation to the solution. This shows that numerics are not always reliable because we make approximations in each step, and this equation is too bad to use linear approximation.
7. (10 points) Solve the difference equation

\[ y_{n+1} = \frac{1}{2} y_n + 1, \quad n \geq 0, \]

in terms of the initial value \( y_0 = 1 \). Describe the behavior of the solution as \( n \to \infty \).

\textit{Solutions.}

\[ y_n = \frac{1}{2^n} \cdot 1 + \frac{1}{2^{n-1}} + \cdots + \frac{1}{2} + 1 \]

\[ = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} \]

\[ = 2 - 2^{-n}. \]

Let \( n \to \infty \) we obtain

\[ \lim_{n \to \infty} y_n = 2 \]
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