Bernoulli equations are a class of first order differential equations of the form

$$y'(t) + p(t)y(t) = q(t)[y(t)]^n$$

where $n \neq 0, 1$. One observes that Bernoulli equations are nonlinear (nonlinearity can be large for large $n$). However, we can make change of variables to convert Bernoulli equations into first order linear equations, where we can use method of integrating factors to find solutions.

First of all, the zero solution $y \equiv 0$ is always a solution to Bernoulli equation. If we now let $y \neq 0$, one can first divide $y^n$ and obtain

$$y^{-n}y' + y^{1-n}p = q.$$ 

Choose change of variable $u = y^{1-n}$ and notice that by chain rule

$$u' = (1 - n)y^{-n}y'.$$

Plugging in the differential equation, we have

$$\frac{1}{1-n}u' + pu = q,$$

or

$$u' + (1 - n)pu = (1 - n)q.$$ 

This is a first order linear equation and we can use method of integrating factors to form the equation into an exact equation.

**Example:** Solve Bernoulli equation:

$$4y' - 2y = -te^{-2t}y^5.$$ 

**Solution:**

If $y \neq 0$, divide equation by $y^5$ we obtain

$$4y^{-5}y' - 2y^{-4} = -te^{-2t}.$$
Let $u = y^{-4}$, the equation is equivalent to

$$u' + 2u = -te^{-2t}.$$ 

Now this equation is a first order linear equation. Integrating factor of this equation is

$$\mu(t) = e^{2t}.$$ 

Forming an exact equation, we obtain

$$(e^{2t}u)' = -t.$$ 

Integrating both sides and substitute back $u = y^{-4}$, we have general (implicit) solution to Bernoulli equation

$$y^4 = \frac{2e^{2t}}{t^2 + C},$$ 

where $C$ is a constant.

Notice that no matter which $C$ we choose, this general solution doesn’t contain the zero solution $y \equiv 0$. By checking that $y \equiv 0$ is also a solution to Bernoulli equation, we conclude that solutions to the equation are

$$[y(t)]^4 = \frac{2e^{2t}}{t^2 + C},$$

and

$$y(t) = 0.$$