§2 Cauchy criterion for integrability in terms of Riemann sums

Recall: A sequence \( \{a_n\}_{n=1}^{\infty} \) is Cauchy if \( \forall \varepsilon > 0, \exists N \in \mathbb{N} \) s.t.

\[
|a_n - a_m| < \varepsilon \quad \forall \ m, n > N
\]

Cauchy criterion for convergence of seq.

If \( \{a_n\}_{n=1}^{\infty} \) is convergent if and only if \( \{a_n\}_{n=1}^{\infty} \) is Cauchy.

\[
\alpha, \beta \quad \xrightarrow{\text{Cauchy}} \quad \mathbb{R}
\]

Theorem (Cauchy criterion for integrability in terms of Riemann sums) Given \( f: [a,b] \to \mathbb{R} \), then \( f \in R(a,b) \) if and only if \( \forall \varepsilon > 0, \exists \lambda > 0 \) s.t.

\[
|S^b_{\lambda}(f) - S^a_{\lambda}(f)| < \varepsilon
\]

for any two Riemann sums \( S^b_{\lambda}(f) \) and \( S^a_{\lambda}(f) \)
with $\delta_1, \delta_2 < \lambda$.

Idea of proof:

- Two directions

- $(\Rightarrow)$ How to find this $\lambda$?

  We know $f \in \mathcal{R}(a,b) \Rightarrow S_{\delta_1}(f) \to S(f)$ as $\delta \to 0$

  \[
  \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall \delta \leq \delta_2 \text{ with } \delta \in \mathbb{R}
  \]

  \[
  |S_{\delta_1}(f) - S(f)| < \varepsilon 
  \]

  \[
  |S_{\delta_1}(f) - S_{\delta_2}(f)| = |S_{\delta_1}(f) - S(f) + S(f) - S_{\delta_2}(f)| 
  \]

  \[
  \leq |S_{\delta_1}(f) - S(f)| + |S_{\delta_2}(f) - S(f)| < 2\varepsilon 
  \]

  as long as $\delta_1, \delta_2 < \lambda$

- $(\Leftarrow)$ Want to use definition to show $f \in \mathcal{R}(a,b)$

  want to show $S_{\delta}(f) \to S(f)$

  Step 1: find $S(f)$ (use completeness of real line)

  Step 2: show $S_{\delta}(f) \to S(f)$
(another use of triangle inequality)

Proof. \((\Rightarrow)\). Since \(f \in \mathcal{R}_{a,b}\), then we know
\[
\int_a^b f(x) \, dx = \frac{1}{n} \sum_{i=1}^{n} f(x_i) \Delta x_i
\]
for all \(\lambda > 0\) such that \(\delta_1, \delta_2 < \lambda\),
\[
\left| \int_{\delta_1}^{\delta_2} f - \int f \right| < \frac{\lambda}{2}
\]
and
\[
\left| \int_{\delta_2}^{\delta_3} f - \int f \right| < \frac{\lambda}{2}
\]
then by triangle inequality,
\[
\left| \int_{\delta_1}^{\delta_3} f - \int f \right| < \left( \left| \int_{\delta_1}^{\delta_2} f - \int f \right| + \left| \int_{\delta_2}^{\delta_3} f - \int f \right| \right)
\leq \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda
\]

\((\Leftarrow)\). Step 1: For simplicity, we assume
\([a,b] = [0,1]\), otherwise, use linear transformation to the partition points to move \([0,1]\) to \([a,b]\).
Take the uniform partition:
\[
P_n = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} = 1 \right\}
\]
We know that as \( n \to \infty \), \( \frac{1}{n} \to 0 \)
we denote the Riemann sums as \( \{S_{\frac{1}{n}}(f)\}_{n=1}^{\infty} \)
when \( n > N := \lceil \frac{1}{\lambda} \rceil \) then \( \frac{1}{n} < \lambda \)
\[
|S_{\frac{1}{n}}(f) - S_{\frac{1}{m}}(f)| < \varepsilon \quad \forall \quad m,n > N
\]
This tells us that \( \{S_{\frac{1}{n}}(f)\}_{n=1}^{\infty} \) is a
Cauchy sequence. Then by completeness
of the real line, we know this sequence
converges, denote the limit as \( S(f) \).

**Step 2:** Want to show \( |S_\delta(f) - S(f)| \to 0 \)
as \( \delta \to 0 \) \& partitions \( P_\delta \)
WTS \( \forall \varepsilon > 0 \exists \eta > 0 \) st. for all
Riemann sums \( S_\delta(f) \) with \( \delta < \eta \),
\[
|S_\delta(f) - S(f)| < \varepsilon
\]
We know \( S_{\frac{1}{n}}(f) \to S(f) \)
\[
|S_\delta(f) - S_{\frac{1}{n}}(f) + S_{\frac{1}{n}}(f) - S(f)| < \varepsilon
\]
By step 1, we know $S_n(f) \to S(f)$ as $n \to \infty$, then $\forall \varepsilon > 0 \exists N > 0 \text{ s.t. for all Riemann sums } S_n(f)$ with $n > N$

$|S_n(f) - S(f)| < \frac{\varepsilon}{2}$

By assumption, $\exists \lambda > 0 \text{ s.t. whenever } S_\delta(f)$ with $\delta < \lambda$

$|S_\delta(f) - S_n(f)| < \frac{\varepsilon}{2}$

as long as $\frac{1}{n} < \lambda$

as long as $n > \max \{N, \frac{1}{\lambda}\}$ then

$|S_\delta(f) - S(f)| = |S_\delta(f) - S_n(f) + S_n(f) - S(f)|$

$\leq |S_\delta(f) - S_n(f)| + |S_n(f) - S(f)|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Then by definition of Riemann integration $\int_a^b f(x) \, dx$
§3 Cauchy criterion for integrability
in terms of Darboux sums

Idea: Want to fix evaluation points.

Def’n (Upper and lower Darboux sums)

Given a partition $P_\delta$, for each $i = 1, 2, \ldots, N$, we let

$$M_i := \sup_{x \in [x_{i-1}, x_i]} f(x), \quad m_i := \inf_{x \in [x_{i-1}, x_i]} f(x)$$

then we denote the upper and lower Darboux sums as

$$U_\delta (f) = \sum_{i=1}^{N} M_i \, (x_i - x_{i-1})$$

and

$$L_\delta (f) = \sum_{i=1}^{N} m_i \, (x_i - x_{i-1})$$
Remark: Upper and Lower Darboux sums bounds the Riemann sums.
Def'n (Upper and lower Darboux integrals)

The upper and lower Darboux integrals are defined as

\[ U(f) = \inf_{\delta > 0} U_\delta (f) \quad \left( = \lim_{\delta \to 0} U_\delta (f) \right) \]

and

\[ L(f) = \sup_{\delta > 0} L_\delta (f) \quad \left( = \lim_{\delta \to 0} L_\delta (f) \right) \]

Theorem (Cauchy criterion for integrability in terms of Darboux sums) Given a function \( f : [a,b] \to \mathbb{R} \), then \( f \in \mathcal{R}(a,b) \) if and only if \( U(f) = L(f) \).

Proof postponed.
Example (A function $f(x)$ that is not Riemann integrable) We define the Dirichlet function as

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q} \end{cases}$$

We show $f(x) \notin R([0, 1])$

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Draw the graph
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Proof. For each partition $P = \{I_i\}$ of $[0, 1]$

By density of rational number and irrational number on the real line,

- $M_i = 1$ and $m_i = 0$

So,

$$L(f) = \frac{\Sigma}{\Sigma} m_i (X_i - X_{i-1}) = 0$$

$$U(f) = \frac{\Sigma}{\Sigma} M_i (X_i - X_{i-1}) = 1$$
so \( \text{L}(f) = 0 \) and \( \text{U}(f) = 1 \)

By Cauchy criterion for integrability in terms of Darboux sums,

\[ f \in \mathcal{R}(a,b) . \]

Def’n (The refinement of partition) Let \( P_0^1 \) and \( P_0^2 \) be two partitions of \([a,b]\), we say \( P_0^2 \) is a refinement of \( P_0^1 \) if \( P_0^1 \subset P_0^2 \).

Example: \([a,b] = [0,1]\)

\[ P_0^1 = \{0, \frac{1}{2}, 1\} \quad P_0^1 = \{0, \frac{1}{3}, \frac{2}{3}, 1\} \]

\[ P_0^3 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\} \]

\( P_0^3 \) is a refinement of \( P_0^1 \).
However, \( P_{\delta_1} \cup P_{\delta_2}^2 = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\} \)

is a refinement of both \( P_{\delta_1}^1 \) and \( P_{\delta_2}^2 \).

**Proposition:** Let \( P_{\delta_1}^1 \) and \( P_{\delta_2}^2 \) be two partitions of \([a,b]\), then
\[
P_{\delta_1}^1 \subset (P_{\delta_1}^1 \cup P_{\delta_2}^2)
\]
and
\[
P_{\delta_2}^2 \subset (P_{\delta_1}^1 \cup P_{\delta_2}^2)
\]
Lemma. If \( f: [a, b] \rightarrow \mathbb{R} \) is bounded and \( P_1, P_2 \) are two portions of \([a, b]\) such that \( P_1 \subset P_2 \), then
\[
L_{\delta, 1}(f) \leq L_{\delta, c}(f) \leq U_{\delta, c}(f) \leq U_{\delta, 1}(f)
\]

Proof. Suppose \( P_1 = \{ a = x_0 < x_1 < \ldots < x_N = b \} \) and we just assume there is only one new endpoint added to \([x_{k-1}, x_k]\) to get \( P_2 \), i.e.
\[
P_2 = \{ a = x_0 < x_1 < \ldots < x_{k-1} < x^* < x_k < \ldots < x_N = b \}
\]

Then
\[
L_{\delta, c}(f) - L_{\delta, 1}(f) = \inf_{x \in [x_{k-1}, x^*]} f(x) \cdot (x^* - x_{k-1}) + \inf_{x \in [x^*, x_k]} f(x) \cdot (x_k - x^*) - \inf_{x \in [x_{k-1}, x_k]} f(x) \cdot (x_k - x_{k-1})
\]
Since \([x_{k-1}, x^*] \subset [x_{k-1}, x_k]\)
and \([x^*, x_k] \subset [x_{k-1}, x_k]\)

then \(\inf_{x \in [x_{k-1}, x^*]} f(x) \geq \inf_{x \in [x_{k-1}, x_k]} f(x)\)

\(\inf_{x \in [x^*, x_k]} f(x) \geq \inf_{x \in [x_{k-1}, x_k]} f(x)\)

then \(L_{\delta_2}(f) - L_{\delta_1}(f)\)

\[\geq \inf_{x \in [x_{k-1}, x_k]} f(x) (x^* - x_{k-1} + x_k - x^*)\]

\[- \inf_{x \in [x_{k-1}, x_k]} f(x) (x_k - x_{k-1})\]

\[= 0\]

\[\Rightarrow L_{\delta_1}(f) \leq L_{\delta_2}(f)\]

Similarly, \(U_{\delta_1}(f) \geq U_{\delta_2}(f)\)
HW: Problem 3