MAT 125B Review

Chapter 1: Riemann integration

- Definition: \( f : [a,b] \to \mathbb{R} \)
  
  \( P_{\delta} = \{ a = x_0 < x_1 < x_2 < \ldots < x_n = b \} \) partition

  \( \epsilon \); evaluation points.

  Riemann sum: \( \mathcal{R}_\delta(f) = \sum_{i=1}^{n} f(\xi_i) (x_i - x_{i-1}) \)

  \( \int_a^b f(x) \, dx = \lim_{\delta \to 0} \mathcal{R}_\delta(f) \) if exists.

  \( f \in R(a,b) \)

- Criterion for integrability.

  - Cauchy criterion for integrability in terms of Riemann sums.

  \( f \in R(a,b) \) iff \( \forall \delta > 0, \exists \lambda > 0 \) st.

  \( |\mathcal{R}_{\delta_1}(f) - \mathcal{R}_{\delta_2}(f)| < \epsilon \)

  whenever \( \delta_1, \delta_2 < \lambda \).

  “\( \frac{\epsilon}{2} \)” - trick
- Cauchy criterion for integrability in terms of Darboux sums.

\( f \in \mathcal{R}(a, b) \iff U(f) = L(f) \)

where \( U(f) = \inf_{\delta > 0} U_\delta(f) \)

\[
= \inf_{\delta > 0} \sum_{i=1}^{N} \sup_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1})
\]

\( L(f) = \sup_{\delta > 0} L_\delta(f) \)

\[
= \sup_{\delta > 0} \sum_{i=1}^{N} \inf_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1})
\]

\( \iff \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } U_\delta(f) - L_\delta(f) < \varepsilon \)

\[
= \sum_{i=1}^{N} \left[ \sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right] (x_i - x_{i-1})
\]

\[
\text{osc}_f \quad [x, x_f]
\]
Examples:
- Continuous functions are integrable.
- Bounded functions with finitely many discontinuities are integrable.
- Monotonic functions are integrable.
- Dirichlet function:
  \[ f(x) = \begin{cases} 
  1 & \text{if } x \in \mathbb{Q} \\
  0 & \text{if } x \not\in \mathbb{Q}
  \end{cases} \]
  is not integrable.
- Riemann function:
  \[ f(x) = \begin{cases} 
  \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ reduced fraction} \\
  0 & \text{if } x \not\in \mathbb{Q}
  \end{cases} \]
  is integrable.
- Unbound functions are not integrable.

Properties of integrable functions
- Linearity
- \( f, g \in \text{Real} \Rightarrow \int_a^b f(x)g(x) \, dx \) exists.
- Monotonicity: if \( f \leq g \) then\[
S^b_a f(x) \, dx \leq S^b_a g(x) \, dx
\]

- Additivity: \( f \in \mathcal{R}(a,b) \), \( c \in [a,b] \)
\[
S^b_a f(x) \, dx = S^c_a f(x) \, dx + S^b_c f(x) \, dx
\]

- Mean value theorem for integrals.
  If \( f \in \mathcal{C}([a,b]) \), then \( \exists x_0 \in [a,b] \) s.t.
  \[
  f(x_0) = \frac{1}{b-a} \int_a^b f(x) \, dx
  \]

- Absolute property: if \( f \in \mathcal{R}(a,b) \), then
  \[
  |\int_a^b f(x) \, dx| \leq \int_a^b |f(x)| \, dx
  \]

- Interchange limits with integrals.
  If \( f \) and \( f_n \) are integrable on \([a,b]\), and \( f_n \rightarrow f \) uniformly, then \( f \in \mathcal{R}(a,b) \).
  Moreover,
  \[
  \lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b \lim_{n \to \infty} f_n(x) \, dx = \int_a^b f(x) \, dx
  \]
If \( f_n \to f \) uniformly if

\[ \forall \varepsilon > 0, \exists N > 0 \text{ s.t. whenever } n > N, \]

we have

\[ |f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in [a, b] \]

- Fundamental theorem of Calculus.
  (i) If \( f \in \mathbb{R}(a,b) \) and \( f \) has an antiderivative \( F \) \((F:[a,b] \to \mathbb{R}, \text{ \( F \) is continuous, and is differentiable on } (a,b), \text{ \( F'(x) = f(x) \)) then}

\[ \int_a^b f(x) \, dx = F(b) - F(a) \]

(ii) If \( g \in \mathbb{R}(a,b) \), then define

\[ G(x) = \int_a^x g(t) \, dt. \]

Then \( G \) is continuous on \([a,b]\). Moreover if \( g \) is continuous at \( x_0 \in [a,b] \), then \( G \) is differentiable at \( x_0 \), \( G'(x_0) = g(x_0) \)
• Integration by parts (integral version of the product rule)

\[ f(x) g(x) = \int_a^b f(x) g(x) \, dx \]

\[ \int_a^b f(x) g(x) \, dx = \int_a^b f(x) g(x) \, dx + \int_a^b f'(x) g(x) \, dx \]

\[ \int_a^b f(x) g(x) \, dx = \left[ f(x) g(x) \right]_a^b - \int_a^b f(x) g'(x) \, dx \]

• Change of variable formula (integral version of the chain rule)

\[ (g \circ f)(x) = g'(f(x)) \cdot f'(x) \]

\[ \int_a^b (g \circ f)(x) \, dx = \int_{f(a)}^{f(b)} g(y) \, dy \]

• Improper integrals

- Definitions (idea: truncate to Riemann integrals and then send to limit)
- Absolute convergence and conditional convergence.
- Principal value integrals
  "Use cancellation to make nonconvergent improper integral converge."

Chapter 2: Differentiable mappings
- Definition of mappings, linear transformations
  - Matrix representation of linear transformations
- Differentiability
  \[ f : \mathbb{A} \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad f \text{ is differentiable at } \mathbf{x}_0 \in \mathbb{A} \text{ if } \exists \varepsilon > 0, \exists \delta > 0 \text{ s.t.} \]
  \[ \| \mathbf{x} - \mathbf{x}_0 \| < \delta, \quad \mathbf{x} \in \mathbb{A} \quad \text{imply} \]
  \[ \| f(\mathbf{x}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \| < \varepsilon \| \mathbf{x} - \mathbf{x}_0 \| \]
  for some linear transformation \( Df(\mathbf{x}_0) \).
- Jacobian matrix and partial derivatives
- Directional derivatives
- has continuous partial derivatives
- differentiable.
- continuous partial derivatives exist
- directional derivatives exist.

• Properties of differentiable mappings
  - chain rule
  - product rule.
  - mean value theorem
    - convex combination \rightarrow convex set.
    - "Differentiable mapping on an open convex set with bounded gradient is Lipschitz."
  - Higher-order derivatives, \( f: \mathbb{R}^n \rightarrow \mathbb{R}^m \)
    \[ D^k f : A \rightarrow \mathbb{R}^{m \times k} \]
    \[ D^k f (x) : (\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \rightarrow \mathbb{R}^m) \text{ \underbrace{\text{k-copies}}_{\text{multilinear form}}} \]
- symmetry of Hessian
- Taylor's theorem and inverse function theorem

HW → evaluation.