HW 2 Solutions

Shawn Witte

Last updated July 4, 2017

5.4.18

Every point in this domain is isolated except for 0, which is an accumulation point. So, by definition 5.33, any function will be continuous at every point except maybe 0. So let \( \{y_n\} \) and \( \{z_n\} \) such that for some \( L \), \( y_n \to L \) and \( z_n \to L \) as \( n \to \infty \) define \( f(1/2^n) = y_n, f(-1/2^n) = z_n, \) and \( f(0) = L \).

To see this, let \( \varepsilon > 0 \), then choose \( N_1, N_2 \) such that for all \( n \geq N_1, |y_n - L| < \varepsilon \) and for all \( n \geq N_2, |z_n - L| < \varepsilon \). Choose \( N = \max\{N_1, N_2\} \) then for all \( x \) such that \( |x - 0| = |x| < \delta = 1/2^N \) we have \( |f(x) - L| < \varepsilon \).

5.5.1

No. For one example, let \( f(x) \) be the characteristic function of \( \mathbb{Q} \) and let \( g(x) \) be the characteristic function of \( \mathbb{R} \setminus \mathbb{Q} \). Then neither \( f(x) \) or \( g(x) \) are continuous, but \( (f + g)(x) = 1 \) for all \( x \), which is continuous.

5.5.4

Let \( f(x) \) be the characteristic function of \( \mathbb{Q} \). Then \( f \) is not continuous anywhere, but \( f(f(x)) = 1 \) for all \( x \), which is continuous.

5.6.3

\( f(x) = \sin(1/x) \) is continuous on \( (0,1) \), but is not uniformly continuous. Let \( \varepsilon = 1/2 \) and let \( \delta > 0 \), then choose \( x_0 = \frac{2}{(2n+1)\pi} \) where \( 2n+1 \) is large enough such that \( |x_0 - 0| < \delta \). Then \( |f(x_0) - f\left(\frac{2}{(2n+3)\pi}\right)| = 2 > \varepsilon \).

5.6.9

Suppose \( f(x) \) is Lipschitz continuous with Lipschitz constant \( M \). Let \( \delta = \varepsilon/M \), then for any \( x, y \) such that \( |x - y| < \delta \), we have

\[
|f(x) - f(y)| \leq M|x - y| < M\delta = M \frac{\varepsilon}{M} = \varepsilon
\]

Therefore \( f \) is uniformly continuous.

The converse is not true. For example, let \( f(x) = \sqrt{x} \), then for each \( \varepsilon > 0 \) choose \( \delta \) such that for any \( x \) in the domain \( (0, \infty) \) satisfying \( |x - 0| < \delta \) implies \( |f(x) - f(0)| = |f(x)| < \varepsilon/2 \). Then for any \( x, y \in (0, \infty) \) such that \( |x - y| < \delta \), we have

\[
|f(x) - f(y)| < |f(x)| + |f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon
\]

due to \( f(x) \) is uniformly continuous.

However, let \( M > 0 \), then choose \( x < 1 \) so that \( \sqrt{x} > x \). Then

\[
|f(M^2x) - f(0)| = \sqrt{M^2x} = M\sqrt{x} > Mx = M|x - 0|
\]

So \( f \) is not Lipschitz continuous.
5.7.5

Case 1: suppose \( f(x) = 0 \) for all \( x \). Then its absolute max and min are 0 which are achieved everywhere.

Case 2: suppose for some \( x_0 \), \( f(x_0) \neq 0 \). Choose \( \varepsilon = \frac{|f(x_0)|}{2} \) then there is some \( a < x_0 \) such that for all \( x < a \), \( |f(x) - 0| = |f(x)| < \varepsilon < |f(x_0)| \), and there is some \( b > x_0 \) such that for all \( x > b \), \( |f(x) - 0| = |f(x)| < \varepsilon < |f(x_0)| \). Hence, for all \( x \not\in [a, b] \), \( |f(x)| > f(x) \), thus at least one of the supremum or infimum of \( f \) on \( \mathbb{R} \) is not achieved in \( \mathbb{R} \setminus [a, b] \).

By theorem 5.50, \( f(x) \) achieves an absolute max \( M \) and min \( m \) over the set \( [a, b] \). We have at least one of \( |M| > |f(x_0)| \) or \( |m| > |f(x_0)| \), thus at least one of \( M \) or \( m \) is the max or min on the entire real line.

5.8.6

The statement is equivalent to the statement that \( f(c) - c = 0 \) for some \( c \in [a, b] \). If \( f(a) = a \) or \( f(b) = b \), then we are done. If this is not the case, then \( f(a) > a \) and \( f(b) < b \), hence \( f(a) - a > 0 \) and \( f(b) - b < 0 \). So, since \( f \) is continuous and must satisfy the Darboux property, there is some \( c \in [a, b] \) where \( f(c) - c = 0 \).