7.2.5

Proof. First note that

\[
f(x) = \begin{cases} 
  x^p & x > 0 \\
  (-x)^p & x < 0 \\
  0 & x = 0 
\end{cases}
\]

Case (i) \( p > 1 \), \( \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{f(h)}{h} = \lim_{h \to 0^+} h^{p-1} = 0 \). Since in this case \( p - 1 \geq 0 \), the function \( f(x) = x^{p-1} \) is continuous at \( x = 0 \), so the limit is 0. Also, \( \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{f(h)}{h} = \lim_{h \to 0^-} \frac{(-h)^p}{h} = \lim_{h \to 0^-} -(-h)^{p-1} = 0 \) Since both the right and left hand limits exist, then \( f \) is differentiable at \( x = 0 \).

Case (ii) \( 0 < p < 1 \): \( \lim_{h \to 0^+} \frac{h^{p-1}}{h} = \lim_{h \to 0^+} \frac{1}{h^{1-p}} = +\infty \). Therefore the limit DNE so \( f \) is not differentiable at 0.

Case (iii) \( p=1 \): We observe the \( \lim_{h \to 0^-} \frac{h^{p-1}}{h} = \lim_{h \to 0^-} 1 = 1 \) and \( \lim_{h \to 0^-} h^{p-1} = \lim_{h \to 0^-} -1 = -1 \) therefore, since the right-hand derivative is not equal to the left-hand derivative, \( f(x) \) is not differentiable at \( x = 0 \).

7.2.15

Proof. Because \( f(x + 0) = f(x) f(0) \), we have \( f(x) = f(x) f(0) \) for all \( x \in \mathbb{R} \). So either \( f(x) = 0 \) for all \( x \in \mathbb{R} \), or \( f(0) = 1 \).

If \( f(x) = 0 \) for all \( x \in \mathbb{R} \), then we have \( f'(0) = 0 \), which contradicts that \( f'(0) = 1 \). So \( f(0) = 1 \).

By the definition of derivative, we have

\[
f'(0) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x}
\]

(1)

So

\[
f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (2)
\]

\[
= \lim_{\Delta x \to 0} \frac{f(x) f(\Delta x) - f(x)}{\Delta x} \quad (3)
\]

\[
= \lim_{\Delta x \to 0} \frac{f(x) (f(\Delta x) - 1)}{\Delta x} \quad (4)
\]

\[
= f(x) \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x} \quad (5)
\]

\[
= f(x) f'(0) \quad (6)
\]

\[
= f(x) \cdot 1 = f(x) \quad (7)
\]
7.2.16

For \( n \) even \( n \geq 2 \), define \( f(x) = 1/n \) on \( \left[ \frac{1}{n}, \frac{1}{n-1} \right] \) and for \( n \) odd \( n > 1 \), define \( f(x) = \frac{2n}{n+1}(x-\frac{1}{n-1}) + \frac{1}{n-1} \) on \( \left[ \frac{1}{n}, \frac{1}{n-1} \right] \). Finally, define \( f(0) = 0 \).

Since \( f(x) \) is simply a line on each segment \( \left[ \frac{1}{n}, \frac{1}{n-1} \right] \), it is continuous except perhaps at \( 1/n \) for \( n = 2, 3, 4, \ldots \).

So, let \( n \) be odd, then

\[
\lim_{x \to \frac{1}{n}^-} f(x) = \lim_{x \to \frac{1}{n}^+} \frac{1}{n} = \frac{1}{n} = \lim_{x \to \frac{1}{n}^-} \frac{2(n+1)}{n+2}(x-\frac{1}{n}) + \frac{1}{n} = \lim_{x \to \frac{1}{n}^-} f(x)
\]

(8)

When \( n \) is even, then

\[
\lim_{x \to \frac{1}{n}^-} f(x) = \lim_{x \to \frac{1}{n}^+} \frac{1}{n+1} = \frac{1}{n+1} = \lim_{x \to \frac{1}{n}^-} \frac{2(n+1)}{n+2}(x-\frac{1}{n-1}) + \frac{1}{n-1} = \lim_{x \to \frac{1}{n}^-} f(x)
\]

(9)

So \( \lim_{x \to 1/n} = f(1/n) \) for all \( n \). Also, note that as \( n \to \infty \) \( f(x) \to 0 \), hence \( \lim_{x \to 0^+} f(x) = 0 = f(0) \). So \( f(x) \) is continuous on all of \([0, 1]\).

However, at each point where \( n \) is odd:

\[
\lim_{x \to \frac{1}{n}^-} \frac{f(x) - f(1/n)}{x - 1/n} = \lim_{x \to \frac{1}{n}^+} 0 = \frac{1}{n} \neq \frac{2(n+1)}{n+2} = \lim_{x \to \frac{1}{n}^-} \frac{f(x) - f(1/n)}{x - 1/n}
\]

(10)

Therefore \( f \) is not differentiable at \( 1/n \) for \( n \) odd. We could prove a similar statement for \( n \) even, but since there are infinitely many points in \([0, 1]\) of the form \( 1/n \) with \( n \) odd, we have already shown that \( f \) is non-differentiable at infinitely many points.

7.3.4

Using the product rule from theorem 7.7:

\[
\frac{d}{dx} (f(x))^2 = (ff')(x) = f(x)f'(x) + f'(x)f(x) = 2f(x)f'(x).
\]

(11)

Alternatively, using the definition,

\[
(ff')(x_0) = \lim_{x \to x_0} \frac{[f(x)]^2 - [f(x_0)]^2}{x - x_0} = \lim_{x \to x_0} \frac{f(x) + f(x_0)}{x - x_0} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = 2f(x_0)f'(x_0).
\]

(12)

(the second equality comes from factoring the numerator, and the first limit may be taken since \( f \) differentiable implies \( f \) continuous.)

7.3.5

For \( n = 1 \), notice that

\[
\frac{d}{dx} (x) = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1 = nx^{n-1},
\]

(13)

establishing the rule for \( n = 1 \). For the induction step, assume that

\[
\frac{d}{dx} (x^{n-1}) = (n-1)x^{n-2}.
\]

(14)

Notice that

\[
\frac{d}{dx} (x^n) = \frac{d}{dx} (x \cdot x^{n-1}) = x \frac{d}{dx} (x^{n-1}) + x^{n-1} \frac{d}{dx} (x) = x(n-1)x^{n-2} + x^{n-1} \cdot 1 = nx^{n-1}.
\]

(15)
7.3.11

Proof. Case 1: \( f'(x_0) \) does not exist.

Nothing remains to be proven.

Case 2: \( f'(x_0) \) exists. Now consider a sequence of points \( x_n \in (x_0 - 1/n, x_0 + 1/n) \) such that \( x_n \neq x_0 \) and \( f(x_n) = f(x_0) \). Then \( x_n \to x_0 \) as \( n \to \infty \). Now by the assumption that \( f'(x_0) \) exists, we know that the limit \( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \) exists. By the sequential definition of limits, we know that it must be the case that

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = \lim_{n \to \infty} \frac{0}{x_n - x_0} = 0
\]

(16)

7.3.18

Take \( I(x) \) to be the inverse function, then \( I(\tan(x)) = x \). Given that \( \frac{d}{dx} \tan(x) = \sec^2(x) \) and the chain rule implying that \( I'(f(x)) = \frac{1}{f'(x)} \), we can say that \( I'(\tan(x)) = \frac{1}{\sec^2(x)} \). Using trigonometric identities, this result can also be written as \( I'(\tan(x)) = \frac{1}{\tan^2(x) + 1} \). Let \( t = \tan(x) \), then \( I'(t) = \frac{1}{t^2 + 1} \), which is the derivative of \( \tan^{-1}(t) \).

7.3.21

Proof. Let \( x > 0 \). Let \( f(x) = x^\frac{m}{n} \) and \( g(x) = x^m \). Thus we have \( f(g(x)) = g(f(x)) = x^n \). Then,

\[
\frac{d}{dx} g(f(x)) = \frac{d}{dx} x^n = nx^{n-1}
\]

as we have previously proven. So

\[
g'(x^\frac{m}{n}) f'(x) = nx^{n-1}
\]

(18)

and \( g'(x) = mx^{m-1} \)

We now have

\[
f'(x) = \frac{nx^{n-1}}{m(x^\frac{m}{n})^{m-1}} = \frac{nx^{n-1}}{mx^{n-x^\frac{m}{n}}} = \frac{n}{m} x^{n-1-n+\frac{m}{n}} = \frac{n}{m} x^\frac{m}{n-1-1}. \]

(19)