7.5.1

$f(x) = x^3$ is a standard example of this. Let $(a, b)$ be an interval containing 0, choose any $x \in (a, b)$ such that $x < 0$, then $f(x) < f(0) = 0$, also for any $x \in (a, b)$ with $x > 0$, we have $f(x) > f(0)$. Thus $f$ can’t be a local max or min.

7.5.2

(a) For $x \neq 0$ we have $f(x)$ the composition, sum, and product of differentiable functions. So for $x \neq 0$, we may calculate $f'(x) = 4x^3(2 + \sin(x^{-1})) - x^2 \cos(x^{-1})$.

For $x = 0$, we calculate directly:

\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^4(2 + \sin(x^{-1}))}{x} = \lim_{x \to 0} x^3(2 + \sin(x^{-1})) = 0
\]

since $|x^3(2 + \sin(x^{-1}))| \leq 2|x|^3$ which goes to 0 as $x \to 0$.

(b) Let $x \neq 0$. Then

\[f(x) = x^4(2 + \sin(x^{-1})) \geq x^4(2 - 1) = x^4 > 0 = f(0)\]

Hence $f$ has an absolute minimum at $x = 0$

(c) Choose any neighborhood $(a, b)$ of 0. Then choose $n$ such that $n > \max\{\frac{1}{2n\pi}, \frac{8}{2n\pi}\}$ then

\[f'\left(\frac{1}{2n\pi}\right) = 4\left(\frac{1}{2n\pi}\right)^3 (2 + \sin(2n\pi)) - \left(\frac{1}{2n\pi}\right)^2 \cos(2n\pi) = \left(\frac{1}{2n\pi}\right)^2 \left(\frac{8}{2n\pi} - 1\right) < 0\]

\[f'\left(\frac{1}{(2n+1)\pi}\right) = 4\left(\frac{1}{(2n+1)\pi}\right)^3 (2 + \sin((2n+1)\pi)) - \left(\frac{1}{(2n+1)\pi}\right)^2 \cos((2n+1)\pi) = \left(\frac{1}{(2n+1)\pi}\right)^2 \left(\frac{8}{(2n+1)\pi} + 1\right)\]

which is greater than 0.

7.6.4

Choose $M > 0$ and fix $y \in [0, \infty)$. Then either $f(y) \geq M$ or $f(y) < M$. If $f(y) \geq M$, then since $f'(x) > 0$ for all $x$, we know that $f$ is increasing, so for any $x > y$, $f(x) > M$.

Now consider that for any $x \in [y, \infty)$,

\[f(x) - f(y) \geq c(x - y),\]

since that inequality being violated would imply, by the mean value theorem, that there is some $z$ between $x$ and $y$ such that $f'(z) < c$, which violates our hypothesis. Now choose

\[x = \left(y + \frac{M}{c} + \frac{|f(y)|}{c}\right)\]
then substituting this expression in to (5) yields

\[ f(x) \geq c(y + \frac{M}{c} + \frac{|f(y)|}{c} - y) + f(y) = M + |f(y)| + f(y) \geq M \] (7)

Hence, for any \( M \), there exists an \( x_0 \) (the \( x \) found above) such that for any \( x > x_0 \), \( f(x) > M \), i.e.

\[ \lim_{x \to \infty} f(x) = \infty \] (8)

7.6.6

The image of \( f \) must necessarily be an interval by the Darboux (intermediate value) property, (which \( f \) has because it is a continuous function). Now suppose for some \( x, y \in [a, b] \) where \( x \neq y \) we have \( f(x) = f(y) \), then by Rolle’s theorem, there must be a \( c \in (a, b) \) such that \( f'(c) = 0 \), but this is not true, hence \( f(x) \neq f(y) \), i.e. \( f \) is one-to-one.

7.6.8

To show this is equivalent, we must show that Lipschitz \( \Rightarrow \) bounded derivative, and bounded derivative \( \Rightarrow \) Lipschitz.

Lipschitz \( \Rightarrow \) bounded derivative: Let \( f \) satisfy the Lipschitz condition for some \( M \). Then we may write

\[ \left| \frac{f(x) - f(y)}{x - y} \right| \leq M \] (9)

\[ \lim_{x \to y} \left| \frac{f(x) - f(y)}{x - y} \right| \leq \lim_{x \to y} M \] (10)

\[ |f'(y)| \leq M \] (11)

for all \( y \in (a, b) \).

Bounded derivative \( \Rightarrow \) Lipschitz: suppose \( |f'(x)| \leq M \) for some \( M > 0 \) and all \( x \in (a, b) \). Now, suppose that there exists some \( x, y \in (a, b) \) such that

\[ \left| \frac{f(x) - f(y)}{x - y} \right| = K > M \] (12)

then there must be a \( c \in (a, b) \) such that \( |f'(c)| = K > M \) which contradicts that the derivative is bounded by \( M \).

7.6.9

(a) Let \( x, y \in [a, b] \), then by the mean value theorem, there is a point \( c \in (a, b) \) such that \( f'(c) \) is equal to the slope of the chord between those points. Hence \( m \in S \Rightarrow m \in D \), i.e. \( S \subseteq D \).

(b) Consider \( f(x) = x^3 \) on \([-1, 1]\). \( f'(0) = 0 \), so \( 0 \in D \), however for \( 0 \) to be in \( S \), there must be \( x \neq y \) such that \( f(x) = f(y) \), however \( x^3 = y^3 \Rightarrow x = y \).

7.7.5

Let \( 0 < x < y \). Suppose \( f(0) = 0 \) and \( f'(x) \) is an increasing function. Note that, from the mean value theorem on \([0, x]\), there is a \( c \in (0, x) \) such that

\[ f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} \] (13)
since $f'(x)$ is increasing, then it must be the case that
\[ f'(x) > f'(c) = \frac{f(x)}{x} \quad (14) \]
Now, note that
\[ \frac{d}{dx} f(x) = \frac{xf'(x) - f(x)}{x^2}. \quad (15) \]
The denominator of this expression is necessarily positive, so \( \frac{d}{dx} f(x) \) is positive if \( xf'(x) - f(x) > 0 \), which is equivalent to (14). Hence \( \frac{f(x)}{x} \) is an increasing function, i.e.
\[ \frac{f(x)}{x} < \frac{f(y)}{y} \quad (16) \]

7.9.7

Let
\[ f(x) = \begin{cases} x^2 \sin(1/x) + 2x^3 + 3x & x \neq 0 \\
0 & x = 0 \end{cases} \quad (17) \]
It has already been shown that
\[ g(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\
0 & x = 0 \end{cases} \quad (18) \]
is differentiable on \( \mathbb{R} \). Then since \( f(x) \) is the sum of \( g(x) \) and a polynomial, then it is the sum of differentiable functions, and is hence differentiable, and
\[ f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) + 6x^2 + 3 & x \neq 0 \\
0 & x = 0 \end{cases} \quad (19) \]
note that
\[ |2x \sin(1/x)| \leq |2x| \quad (20) \]
\[ |\cos(1/x)| \leq 1. \quad (21) \]
Consider for \( |x| > 1 \),
\[ |2x| < 6x^2 \quad (22) \]
\[ 1 < 3 \quad (23) \]
so whether or not \( 2x \sin(1/x) \) or \( \cos(1/x) \) take negative values, the expression \( 2x \sin(1/x) - \cos(1/x) + 6x^2 + 3 > 0 \). Now consider \( |x| < 1 \):
\[ |2x \sin(1/x)| \leq 1 \quad (24) \]
\[ |\cos(1/x)| \leq 16x^2 > 0. \quad (25) \]
which means \( 2x \sin(1/x) - \cos(1/x) + 6x^2 + 3 > 0 \). So \( f'(x) > 0 \) for all \( x \neq 0 \), and \( f'(0) = 0 \), so \( f \) is non-decreasing, but we note that \( \lim_{x \to 0} f'(x) \) does not exist, so \( f'(x) \) is not continuous.

7.12.2

Let \( f(x) = \sum_{k=0}^{n} a_k x^k \). Then for all \( j > n \), \( f^j(x) = 0 \), hence
\[ f(x) = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k = \sum_{k=0}^{n} \frac{f^k(0)}{k!} x^k \quad (26) \]
So it remains to show that

\[ a_k = \frac{f^k(0)}{k!}. \tag{27} \]

Note that

\[ \frac{d^j}{dx^j} \sum_{k=0}^{n} a_k x^k = \sum_{k=j}^{n} \frac{k!}{(k-j)!} a_k x^{k-j} \tag{28} \]

so all coefficients \( a_k \) vanish for \( k < j \), and when 0 is substituted for \( x \), all of the coefficients vanish for \( k > j \), leaving \( f^j(0) = j! a_j \), which is equivalent to (27).