HW 7 Solutions
Shawn Witte
Last updated August 2, 2017

1 13.2.1

(a) Let $x = 1$, then $d(x,x) = |x| + |x| = 2$, but by definition of a metric, this should be 0, so it fails to be a metric.

(b) Let $x = 1$, $y = 2$, $z = 1.5$. Then $d(x,y) = 1$, but $d(x,z) = d(z,y) = 0.25$, so $1 = d(x,y) > d(x,z) + d(z,y) = 0.5$ violates the triangle inequality. So it fails to be a metric.

(c) This is a metric. Note that it is always nonnegative, and $\sqrt{|x-y|} = 0$ if and only if $x = y$, and also $\sqrt{|x-y|} = \sqrt{|y-x|}$. Now notice that

$$|x-y| = |x-z+z-y| \leq |x-z| + |z-y| \leq |x-z| + 2\sqrt{|x-z||z-y| + |z-y|} = \left(\sqrt{|x-z|} + \sqrt{|z-y|}\right)^2$$

(1)

taking the square root of the left and righthand sides gives us the triangle inequality for this metric.

(d) Note that $|x-y|$ satisfies all requirements of a metric, and with this minimum we still have the first three properties of metrics. Now suppose $d(x,y) < 1$ then $d(x,y) > d(x,z) + d(z,y)$ would imply that both $d(x,z)$ and $d(z,y)$ are smaller than 1, which means they are $|x-z|$ and $|z-y|$, so they must satisfy the triangle inequality, hence $d(x,y) > d(x,z) + d(z,y)$ can’t be true. Now if $d(x,y) = 1$, then the triangle inequality holds if either $d(x,z)$ or $d(z,y)$ is 1. If they are both less than 1, then they must hold the triangle inequality with $|x-y|$ in which case $d(x,y) = 1 \leq |x-y| \leq |x-z| + |z-y|$ so the triangle inequality must hold.

(e) The first three properties of metrics follow form the definition follow from the properties of $|x-y|$. Now notice that if $f(t) = \frac{t}{1+t^2}$ then $f'(t) = \frac{1}{(1+t^2)^2}$ which is always positive, i.e. $f(t)$ is increasing on $\mathbb{R}$. Now consider that $|x-y| \leq |x-z| + |z-y|$, then we have that $f(|x-y|) \leq f(|x-z| + |z-y|)$, or otherwise written as

$$\frac{|x-y|}{1 + |x-y|} \leq \frac{|x-z| + |z-y|}{1 + |x-z| + |z-y|} \quad (2)$$

$$= \frac{|x-z|}{1 + |x-z| + |z-y|} + \frac{|z-y|}{1 + |x-z| + |z-y|} \quad (3)$$

$$\leq \frac{|x-z|}{1 + |x-z|} + \frac{|z-y|}{1 + |z-y|} \quad (4)$$

(f) This was proven to be a metric in class (triangle inequality is proven by cases).

2 13.2.10

$\|x-y\| \geq 0$ for any inputs by properties of norms and $\|x-y\| = 0$ if and only if $x-y = 0$ which is equivalent to $x = y$. We also have $\|x-y\| = \|-1\||x-y|| = \|y-x\|$ by the properties of norms. Finally,

$$\|x-y\| = \|x-z + z-y\| \leq \|x-z\| + \|z-y\|$$

(5)

So $\|x-y\|$ satisfies all the properties of a metric.
3 13.3.7

$\mathcal{A}$ and $\mathcal{S}$ are subspaces of $M(\mathbb{R})$ since they only contain bounded functions, but $\mathcal{P}$ and $\mathcal{C}$ are not, as they contain non-bounded functions on $\mathbb{R}$.

4 13.4.7

Let $x_n \to x$ and $y_n \to y$. Note by applying the triangle inequality twice that

$$d(x, y) \leq d(x, x_n) + d(x_n, y) + d(y, y_n) \leq d(x, y)$$

also

$$d(x, y) \leq d(x, x_n) + d(x, y_n) + d(y, y_n)$$

which yields

$$d(x, y) - d(x, x_n) - d(y, y_n) \leq d(x_n, y_n) \leq d(x, x_n) + d(x, y) + d(y, y_n)$$

taking the limits, we squeeze $d(x_n, y_n)$ so that

$$d(x, y) \leq \lim_{n \to \infty} d(x_n, y_n) \leq d(x, y)$$

hence $\lim_{n \to \infty} d(x_n, y_n) = d(x, y)$

5 13.4.14

(a) $t^k$ does not converge in $C^1[0,1]$ since $kt^{k-1} \to \infty$ for $t = 1$

(b) Note that $t/k$ and its derivative $1/k$ are both bounded by $1/k$ for all $k$.

$$d(x_k, 0) = \max \left| \frac{t}{k} - 0 \right| + \max \left| \frac{1}{k} - 0 \right| = \frac{2}{k}$$

which goes to 0 as $k \to \infty$

(c) $\sin(kt)/k$ does not converge with this metric. Let $y'(\pi/4) = c$ for some $c$. Note that the derivative $x'_k(\pi/4) = \cos(k\pi/4)$ which oscillates between $\pm 1$, $\pm \sqrt{2}/2$ and 0, so it never converges to $c$. That is, $\max |\cos(kt) - y'(t)| \geq |\cos(k\pi/4) - c|$ which is bounded away from zero for some $n \geq N$ for any chosen $N$. 
