125A Midterm 2
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You may not use calculators or any other electronic device during the midterm. Solutions must be written on the blank paper provided. Make sure your name is on each page, and each answer is clearly labeled. You may use any statements proven in class or homeworks (unless it’s the thing you are asked to prove). If you can’t prove something directly, try contradiction.

1 Let \( f: (a, b) \rightarrow \mathbb{R} \) be differentiable at \( c \in (a, b) \) with \( f'(c) > 0 \).

a. Prove that there exists a \( \delta \) such that \( f(x) > f(c) \) for \( x \in (c, c+\delta) \), and \( f(x) < f(c) \) for \( x \in (c-\delta, c) \).

Proof. Since
\[
\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0 \tag{1}
\]
There is some \( \delta \) such that for \( x \in (c-\delta, c+\delta) \),
\[
\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \frac{f'(c)}{2} \tag{2}
\]
which gives us that
\[
0 < \frac{f'(c)}{2} < \frac{f(x) - f(c)}{x - c} \tag{3}
\]
So in order for that fraction to be positive, we need either
\[
f(x) - f(c) < 0 \text{ and } x - c < 0 \tag{4}
\]
or
\[
f(x) - f(c) > 0 \text{ and } x - c > 0 \tag{5}
\]
which can be rewritten as
\[
f(x) < f(c) \text{ and } x < c \tag{6}
\]
and
\[
f(x) > f(c) \text{ and } x > c \tag{7}
\]
\( \square \)

b. Let
\[
f(x) = \begin{cases} 
\frac{x}{2} + x^2 \sin(1/x) & x \neq 0 \\
0 & x = 0
\end{cases} \tag{8}
\]
Show that \( f'(0) = 1/2 \), but \( f'(x) \) takes both positive and negative values on any neighborhood of 0.

Proof.
\[
f'(0) = \lim_{x \to 0} \frac{x/2 + x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} 1/2 + x \sin(1/x) = 1/2 \tag{9}
\]
Now note that for \( x \neq 0 \), \( f'(x) = 1/2 + 2x \sin(1/x) - \cos^2(1/x) \). For any neighborhood of 0, we can choose \( n \) large enough so that \( x_1 = \frac{1}{2n\pi} \) and \( x_2 = \frac{1}{(2n+1)\pi} \) are both inside the neighborhood. Note that \( f'(x_1) = -1/2 \) and \( f'(x_2) = 3/2 \). \( \square \)
2 A function \( f \) is \textit{Lipschitz continuous} on an interval \([a, b]\) if for some constant \( M \), \(|f(x) - f(y)| \leq M|x - y|\) for all \( x, y \in [a, b] \)

a. Prove that if \( f \) is Lipschitz continuous on \([a, b]\) and differentiable on \((a, b)\), then \( f'(x) \) is bounded on \((a, b)\).

\[ \text{Proof.} \text{ Suppose } |f(x) - f(y)| \leq M|x - y|, \text{ then for all } x, y \in (a, b) \]
\[
\begin{align*}
|f(x) - f(y)| & \leq M|x - y| \\
\frac{|f(x) - f(y)|}{|x - y|} & \leq M \\
|f(x) - f(y)| & = |f'(c)||x - y| \leq M|x - y| \\
\lim_{x \to y} \frac{|f(x) - f(y)|}{|x - y|} & \leq \lim_{x \to y} M \\
|f'(y)| & \leq M \\
\end{align*}
\]

b. Prove that if \( f \) is differentiable with \( f'(x) \) bounded on \((a, b)\) and continuous on \([a, b]\), then \( f \) is Lipschitz continuous.

\[ \text{Proof.} \text{ Let } x, y \in [a, b]. \text{ Then by the mean value theorem, there exists a } c \text{ such that} \]
\[
\begin{align*}
\frac{|f(x) - f(y)|}{|x - y|} & = |f'(c)| \\
|f(x) - f(y)| & = |f'(c)||x - y| \leq M|x - y| \\
\lim_{x \to y} \frac{|f(x) - f(y)|}{|x - y|} & \leq \lim_{x \to y} M \\
|f'(y)| & \leq M \\
\end{align*}
\]

c. Find an example of a function which is differentiable, but not Lipschitz continuous.

\[ \text{Proof.} \text{ Let } f(x) = \sqrt{x} \text{ on } [0, 1]. \text{ It is continuous on this interval, and differentiable on } (a, b), \text{ but } f'(x) = \frac{1}{2\sqrt{x}} \text{ goes to infinity as } x \to 0^+. \]

3 Let \( f : \mathbb{R} \to \mathbb{R} \) be the Thomae function (below), prove whether or not \( f \) is differentiable at \( x = 0 \)

\[
f(x) = \begin{cases} 
0 & x = 0 \text{ or } x \in \mathbb{R} \setminus \mathbb{Q} \\
\frac{1}{q} & x = p/q
\end{cases}
\]

where \( p, q \in \mathbb{Z} \) and \( p/q \) is expressed in lowest terms.

\[ \text{Proof.} \text{ Let } x_n = 1/n \text{ and } y_n = \sqrt{2}/n. \text{ Note that } f(x_n) = 1/n \text{ and } f(y_n) = 0. \text{ Then} \]
\[
\begin{align*}
\lim_{n \to \infty} \frac{f(x_n) - 0}{x_n - 0} & = \lim_{n \to \infty} \frac{1/n}{1/n} = \lim_{n \to \infty} 1 = 1 \\
\lim_{n \to \infty} \frac{f(y_n) - 0}{y_n - 0} & = \lim_{n \to \infty} \frac{0}{\sqrt{2}/n} = \lim_{n \to \infty} 0 = 0 \\
\end{align*}
\]

Since these limits do not agree, then by the sequential definition of limits,
\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}
\]

does not exist, i.e. \( f \) is not differentiable at 0
4 Let \( \{f_n\} \) be a sequence of functions \( f_n : \mathbb{R} \to \mathbb{R} \), and \( f : \mathbb{R} \to \mathbb{R} \)

a. Define \( f_n \to f \) uniformly on \( \mathbb{R} \)

\[ \text{Proof.} \quad \text{For every } \varepsilon > 0 \text{ there exists an } N \in \mathbb{N} \text{ such that for all } n \geq N, \text{ and all } x \in \mathbb{R}, \quad |f_n(x) - f(x)| < \varepsilon. \]

b. Give an example of a sequence \( \{f_n\} \) that are bounded, but converge pointwise to something non-bounded.

\[ \text{Proof.} \quad f_n(x) = \begin{cases} |x| & \text{if } |x| < n \\ 0 & \text{if } |x| \geq n \end{cases} \]

\[ |f_n(x)| \leq n, \text{ but the limit is } |x| \text{ which is unbounded on } \mathbb{R} \]

\[ \text{Proof.} \quad \text{Let } \varepsilon = 1, \text{ let } n \in \mathbb{N}. \text{ Choose } x = n, \text{ then} \]

\[ |f_n(x) - f(x)| = |0 - n| = n \geq \varepsilon \]