1. Suppose that a function has both a right-hand and a left-hand derivative at a point. What, if anything, can you conclude about the continuity of that function at that point?

The function must be continuous at that point (let us call it c) because

$$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x) \qquad (*)$$

To prove (*) we observe that

$$\lim_{x \to c^-} (f(x) - f(c)) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}(x - c)$$

$$= \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \lim_{x \to c^-} (x - c) = f_-'(c) \cdot 0 = 0$$

Thus $\lim_{x \to c^-} f(x) = f(c)$.

In a similar way one can show that

$$\lim_{x \to c^+} f(x) = f(c).$$

2. A function $f : I \to R$ is called Lipschitz if there exists a bound $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M$$

for all $x, y \in I$.

(a) Show that if f is Lipschitz on I, then it is uniformly continuous on I.

(b) Is the converse statement true? In other words, are all uniformly continuous functions necessarily Lipschitz?

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a) Let $\varepsilon > 0$ Choose $\delta = \frac{\varepsilon}{M}$

Then $|x-y| < \delta$ imply $|f(x)-f(y)| \leq$

$M|x-y| < M\delta = M\frac{\varepsilon}{M} = \varepsilon$.

Therefore, $f(x)$ is uniformly continuous

b) No. Consider $f(x) = \sqrt{x}$.

$f(x)$ is uniformly continuous on $[0,1]$

(since f is continuous there).

However for $x_n = \frac{1}{n}$, $y_n = 0$, one

has $\frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = \frac{\frac{1}{\sqrt{n}}}{\frac{1}{n}} = \sqrt{n} \to \infty$ as $n \to \infty$

Thus, f is <u>not</u> Lipschitz on $[0,1]$.

3. Let h be a differentiable function on the interval $[0,3]$ and assume that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

(a) Prove that there exists a point $c \in [0,3]$ where $h(c) = c$.

(b) Prove that at some point $d \in [0,3]$ we have $h'(d) = \frac{1}{3}$.

(c) Prove that at some point $x \in [0,3]$ we have $h'(x) = \frac{1}{4}$.

▷ a) Consider $f(x) = h(x) - x$.
Then $f(0) = 1$, $f(3) = -1$
By Intermediate Value Property for f (one can apply it since f is differentiable $\Rightarrow f$ is continuous on $[0,3]$)
$\exists\, c \in [0,3]$ such that $f(c) = 0 \Rightarrow h(c) = c$.

b) $h(3) = 2$, $h(0) = 1$ Apply MVT for h on $[0,3]$. Then $\exists\, d \in [0,3]$ s.t. $h'(d) = \dfrac{h(3) - h(0)}{3 - 0} = \dfrac{1}{3}$.

c) By Rolle's thm, $\exists\, p \in [1,3]$ s.t. $h'(p) = 0$ (since $h(1) = h(3)$).
Then apply Darboux thm for h' on $[d, p]$ to claim $\exists\, x$ s.t. $h'(x) = \frac{1}{4}$.

4. Let $f : R \to R$ be a continuous function with the property

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

Show that f has either an absolute maximum or an absolute minimum but not necessarily both.

Case 1 $\quad f \equiv 0$

Case 2 $\quad f \not\equiv 0 \Rightarrow \exists\, c$ s.t

$|f(c)| \neq 0$. Take $\varepsilon = |f(c)|$

$\exists\, N_1, N_2 > 0$ s.t

$|f(x)| < \varepsilon$ for $x > N_1$

and $|f(x)| < \varepsilon$ for $x < -N_2$.

(here we applied $\lim\limits_{x\to\pm\infty} f(x) = 0$).

Consider $f(x)$ on $[-N_2, N_1]$

If $f(c) > 0$, consider a global maximum of f on $[-N_2, N_1]$. Then it is also a global maximum of f on R.

Similar argument works for global min.

Examples: $f(x) = \dfrac{1}{x^2+1}$; $g(x) = \dfrac{x}{x^2+1}$.