**MATH 125B, HW1, Solution**

5.1.1(b) Since $f(x) = 3 - x^2$ is decreasing on the interval $[0, 2]$, then

$$U(f,P) = f(\frac{1}{2})(\frac{1}{2} - 0) + f(1)(1 - \frac{1}{2}) + f(2)(2 - 1);$$

$$L(f,P) = f(0)(\frac{1}{2} - 0) + f(\frac{1}{2})(1 - \frac{1}{2}) + f(1)(2 - 1).$$

Graph and use geometric interpretation of integrals to explain.

5.1.3 Note that $\lim_{n\to\infty} \frac{1}{n} = 0$. For any $\epsilon > 0$, there exist $N_0 \in \mathbb{N}$, for all $n > N_0$, we have $\frac{1}{n} < \frac{\epsilon}{2}$. Split the set E into two parts, $E_{N_0} = \left\{\frac{1}{n}, n > N_0\right\}$, and $E - E_{N_0} = \left\{\frac{1}{n}, 1 \le n \le N_0\right\}$. All points in $E_{N_0}$ belong to the interval $(0, \frac{\epsilon}{2})$. Define the partition as

$$P = \left\{0, \frac{1}{N_0+1}, \frac{1}{N_0} - \frac{\epsilon}{4N_0}, \frac{1}{N_0} + \frac{\epsilon}{4N_0}, \cdots, \frac{1}{2} - \frac{\epsilon}{4N_0}, \frac{1}{2} + \frac{\epsilon}{4N_0}, 1 - \frac{\epsilon}{2N_0}\right\}.$$

Then

$$U(f,P) - L(f,P) < \frac{\epsilon}{2} + N_0 \times \frac{\epsilon}{2N_0} = \epsilon.$$

**Remark:** The partition P must be a FINITE set.

5.1.5 $\int_a^c f(x)dx = 0$ for all $c \in [a,b]$ implies that for all $[c,d] \subset [a,b]$, we have $\int_c^d f(x)dx = 0$. Assume that there exists some point $x_0 \in (a,b)$, with $f(x_0) \neq 0$. Without loss of generality, suppose $f(x_0) = L > 0$. Since f is continuous at $x = x_0$, for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$, for all $0 < |x - x_0| < \delta$, $|f(x) - L| < \epsilon$. Then on the interval $(x_0 - \delta_\epsilon, x_0 + \delta_\epsilon) - x_0$, we have $L - \epsilon < f(x) < L + \epsilon$. Choose $\epsilon = \frac{L}{2} > 0$, then

$$0 < \frac{L}{2} < f(x) < \frac{3L}{2}.$$

Choose $c = x_0 - \delta_{\frac{L}{2}}$, $d = x_0 + \delta_{\frac{L}{2}}$, then $(L)\int_c^d f(x)dx > \frac{L}{2}(d - c) > 0$. Then we find the contradiction, thus $f(x) = 0$ for all $x \in (a,b)$. Similarly, we can prove $f(a) \neq 0$ and $f(b) \neq 0$.
**Remark:** If $x_0$ =a or b, need to be more careful, the value of x cannot be out of [a,b].

5.1.9 Since $f : [a,b] \to [c,d]$, with $0 < c < d$, then for any $x, y \in [a,b]$,

$$\sqrt{f(x)} - \sqrt{f(y)} = \frac{f(x) - f(y)}{\sqrt{f(x)} + \sqrt{f(y)}}.$$

Note $0 < c \le f(x), f(y) \le d$, then if $f(x) > f(y)$,

$$\sqrt{f(x)} - \sqrt{f(y)} \le \frac{f(x) - f(y)}{2\sqrt{c}}.$$

Since f is integrable on [a,b], this implies that for any $\epsilon > 0$, there exists a partition $P = \{a = x_0, x_1, \cdots, x_n = b, \}$, such that $U(f,P) - L(f,P) = \sum_{k=1}^n (M_k(f) - m_k(f))\Delta_k \le 2\sqrt{c}\epsilon$. Thus

$$U(\sqrt{f}, P) - L(\sqrt{f}, P) = \sum_{k=1}^n (M_k(\sqrt{f}) - m_k(\sqrt{f}))\Delta_k \le \sum_{k=1}^n \frac{(M_k(f) - m_k(f))}{2\sqrt{c}}\Delta_k \le \epsilon.$$

Therefore, $\sqrt{f}$ is integrable.

5.1.10 Use the definition of Riemann Integrable and note that for all partitions, $U(f,P) > L(f,P)$.

5.2.2 (a)Since n is even, $x^n \geq 0$, and f is continuous function, by First Mean Value Theorem for Integrals, there exist $c \in [a, b]$, such that

$$0 = \int_a^b f(x)x^n dx = f(c)\int_a^b x^n dx.$$

Since [a,b] is non-degenerate, then $\int_a^b x^n dx$ is positive, thus $f(c) = 0$.
(b)We can find a counterexample by choosing $f(x) = 1, a = -1, b = 1, n = 3$.

5.2.5 Since f is integrable on [0,1], f must be bounded, say there exist M, such that $f(x) \leq M$. By comparison theorem of integrals,

$$\int_0^{\frac{1}{n^\beta}} f(x)dx \leq \int_0^{\frac{1}{n^\beta}} M dx = \frac{M}{n^\beta}.$$

Therefore, $n^\alpha \int_0^{\frac{1}{n^\beta}} f(x)dx \leq \frac{M}{n^{\beta-\alpha}} \to 0$ as $\beta > \alpha$.

5.2.8 (a)Since $M = sup_{x\in[a,b]}|f(x)|$, then for all $x \in [a, b]$, we have $|f(x)| \leq M$, thus $|f(x)|^p \leq M^p$. By comparison theorem for integrals, we can prove that

$$\int_a^b |f(x)|^p dx \leq M^p(b-a).$$

Also, for any $\epsilon > 0$, there exists $x_0 \in [a, b]$, such that,

$$|f(x_0)| \geq M - \frac{\epsilon}{2}.$$

Since f is continuous at $x = x_0$, so is $|f|$. For any $\epsilon > 0$, there exists $\delta > 0$, for all $0 < |x - x_0| < \delta$,

$$||f(x)| - |f(x_0)|| \leq \frac{\epsilon}{2}.$$

Then on the interval $(x_0 - \delta, x_0 + \delta) - x_0$,

$$|f(x)| \geq |f(x_0)| - \frac{\epsilon}{2} \geq M - \epsilon.$$

By using comparison theorem for integrals, $I = [x_0 - \delta, x_0 + \delta] \subset [a, b]$, and $|f|^p$ is non-negative,

$$\int_a^b |f(x)|^p dx \geq \int_I (M-\epsilon)^p dx = (M-\epsilon)^p|I|.$$

(b) Since

$$(M-\epsilon)^p|I| \leq \int_a^b |f(x)|^p dx \leq M^p(b-a),$$

$$(M-\epsilon)|I|^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \leq M(b-a)^{\frac{1}{p}},$$

then

$$\liminf_p \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \geq M - \epsilon,$$

and

$$\limsup_p \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \leq M.$$

Therefore,

$$\lim_{p\to\infty} \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} = M.$$