MATH 125B, HW2, Solution

5.3.2(a) Change the variable $t = \sqrt{x}$, and $dx = 2tdt$. then

$$\int_1^4 f(\sqrt{x})dx = \int_1^2 f(t)2tdt = 2\int_1^2 f(t)tdt = 12$$

5.3.2(b) Change the variable $t = \frac{1}{x^2}$, and $dx = -\frac{1}{2}t^{-\frac{3}{2}}dt$. then

$$\int_{\frac{\sqrt{2}}{2}}^1 f(\frac{1}{x^2})dx = \int_2^1 f(t)\left(-\frac{1}{2}t^{-\frac{3}{2}}\right)dt = \frac{1}{2}\int_1^2 f(t)t^{-\frac{3}{2}}dt.$$

Since $t \geq 1$, $\int_1^2 f(x)dx = 5$, then

$$\frac{1}{2}\int_1^2 f(t)t^{-\frac{3}{2}}dt \leq \frac{1}{2}\int_1^2 f(t)dt = \frac{5}{2}.$$

5.3.5 Assume $F \in C^1[a,b]$. Define $F' = f$ and f is continous. By the First Mean Theorem for Integrals, choose $g(x) = 1$, since f is continous on $[a,b]$, then there exists an $x_0 \in [a,b]$, such that

$$\int_a^b f(x)dx = f(x_0)(b-a).$$

By the Fundamental Theorem of Caculus, we have

$$F(b) - F(a) = F'(x_0)(b-a).$$

5.3.6 Define the function

$$F(x) = \alpha\int_a^x f(t)dt + \beta\int_x^b f(t)dt = 0$$

Then by Fundamental Theorem of Caculus, for all $x \in [a,b]$,

$$F'(x) = \alpha f(x) - \beta f(x) = (\alpha - \beta)f(x) = 0.$$

Since $\alpha \neq \beta$, then $f(x) = 0$ for all $x \in [a,b]$.

5.3.9 Define $G(x) = xf(x)$, then $G'(x) = f(x) + xf'(x)$. Then by Fundamental Theorem of Caculus, we have

$$\int_a^b G'(x)dx = G(b) - G(a),$$

$$\int_a^b f(x)dx + \int_a^b xf'(x)dx = bf(b) - af(a).$$

Change the variable $f(x) = t$, $x = f^{-1}(t)$, $f'(x)dx = dt$, then

$$\int_a^b xf'(x)dx = \int_{f(a)}^{f(b)} f^{-1}(t)dt.$$

5.4.0 (a) False. To use comparison theorem for improper integrals, we need f to be locally integrable.
(b) True. Since f, g are contiuous, and g is never zero on $[a,b]$, then $\frac{f}{g}$ is continuous on $[a,b]$, thus locally integrable. Since $|g|$ are contiuous and is never zero on $[a,b]$, there exist $0 < c < C$, such that $c \leq |g(x)| \leq C$.Thus,

$$|\frac{f}{g}| \leq \frac{|f|}{c}.$$

Then using Comparison Theorem for Improper Integrals, we can show that $\frac{f}{g}$ is absolute integrable.
(c) True. f is continuous implies that $\sqrt{f}$ is also continuous, thus locally integrable. Since we have the inequality that $\sqrt{f} \leq f + 100$, by using Comparison Theorem for Improper Integrals, we can prove it.
(d) True. $\max\{f, g\} = \frac{|f-g|+f+g}{2}$, and $\min\{f, g\} = \frac{-|f-g|+f+g}{2}$, thus locally integrable. $|\min\{f, g\}| \leq |f|$, and $|\max\{f, g\}| \leq |f| + |g|$.

5.4.3 (a) For $p > 1$,

$$|\frac{\sin x}{x^p}| \leq \frac{1}{x^p}.$$

By using Comparison Theorem for Improper Integrals, $\frac{sinx}{x^p}$ is absolute integrable and thus improper integrable.
For $0 < p \leq 1$, using integration by parts,

$$\int_1^\infty \frac{\sin x}{x^p} dx = -\int_1^\infty \frac{1}{x^p} d\cos x = -\frac{\cos x}{x^p}|_1^\infty + \int_1^\infty \cos x d\frac{1}{x^p}$$

$$= \cos 1 - \lim_{x\to\infty} \frac{\cos x}{x^p} - p\int_1^\infty \frac{\cos x}{x^{p+1}} dx.$$

Since $p + 1 > 1$, then $|\frac{\cos x}{x^{p+1}}|$ is improper integrable on $[1, \infty]$. And $\lim_{x\to\infty} \frac{\cos x}{x^p} = 0$. Therefore, for all $p > 0$, $\frac{\sin x}{x^p}$ is improper integrable on $[1, \infty]$.
(b) Using integration by parts,

$$\int_e^\infty \frac{\cos x}{\log^p x} dx = \int_e^\infty \frac{1}{\log^p x} d(\sin x) = \frac{\sin x}{\log^p x}|_e^\infty + p\int_e^\infty \frac{\sin x}{x\log^{p+1} x} dx$$

$$= \lim_{x\to\infty} \frac{\sin x}{\log^p x} - \sin e + p\int_e^\infty \frac{\sin x}{x\log^{p+1} x} dx.$$

Since $p > 0$, $\lim_{x\to\infty} \frac{\sin x}{\log^p x} = 0$. Also,

$$|\frac{\sin x}{x\log^{p+1} x}| \leq \frac{1}{x\log^{p+1} x}.$$

For the integral above, change the variable $\log x = u$, then

$$\int_e^\infty \frac{1}{x\log^{p+1} x} dx = \lim_{b\to\infty} \int_e^b \frac{1}{x\log^{p+1} x} dx = \lim_{b\to\infty} \int_1^{\log b} \frac{1}{u^{p+1}} du = \int_1^\infty \frac{1}{u^{p+1}} du.$$

Since $p + 1 > 1$, $\frac{\cos x}{\log^p x}$ is improper integrable for $p > 0$.

5.4.5 Choose $f(x) = g(x) = \frac{1}{\sqrt{x}}$ on the interval $(0, 1)$.

5.4.8 Change the variable $t = x^n$, $dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$,

$$\int_1^\infty f(x^n)dx = \lim_{b\to\infty} \int_1^b f(x^n)dx = \lim_{b\to\infty} \int_1^{b^n} f(t)\frac{1}{n} t^{\frac{1}{n}-1} dt = \frac{1}{n}\int_1^\infty f(t)t^{\frac{1}{n}-1} dt.$$

Since $t \leq 1$,

$$\lim_{n\to\infty} \int_1^\infty f(x^n)dx \leq \lim_{n\to\infty} \frac{1}{n} \int_1^\infty f(t)dt.$$

Since f is absolute integrable on $[1, \infty]$, then $\int_1^\infty f(x^n)dx < \infty$, therefore the limit goes to 0.

Extra (a) By Intermediate Value Theorem for Derivatives, for any $x \neq y$, the exists a $c \in (x, y)$, such that

$$\frac{\sin(x) - \sin(y)}{x - y} = \cos(c),$$

thus

$$|\sin(x) - \sin(y)| \leq |x - y|.$$

Thus

$$|\sin(f(x)) - \sin(f(y))| \leq |f(x) - f(y)|. \tag{1}$$

Since f is integrable on the interval $[a, b]$, then for all $\epsilon > 0$, there exsit a parition $P = \{a = x_0, x_1, \cdots, x_n = b\}$, such that

$$\sum_k (M_k(f) - m_k(f))\,\Delta_k < \frac{\epsilon}{2}.$$

For any $\epsilon > 0$, there exists $c_k, d_k \in [x_{k-1}, x_k]$, such that

$$\sin(f(c_k)) \geq M_k(\sin(f)) - \frac{\epsilon}{4(b-a)},$$
$$\sin(f(d_k)) \leq m_k(\sin(f)) + \frac{\epsilon}{4(b-a)}.$$

Thus

$$M_k(\sin(f)) - m_k(\sin(f)) \leq \sin(f(c_k)) - \sin(f(d_k)) + \frac{\epsilon}{2(b-a)} \leq f(c_k) - f(d_k) + \frac{\epsilon}{2(b-a)} \leq M_k(f) - m_k(f) + \frac{\epsilon}{2(b-a)}.$$

Therefore,

$$\sum_k (M_k(\sin(f)) - m_k(\sin(f)))\,\Delta_k \leq \frac{\epsilon}{2} + (b-a) \times \frac{\epsilon}{2(b-a)} = \epsilon.$$

(b) We could find a counterexample $f(x) = x, a = 0, b = \frac{\pi}{2}$. Since $\tan x$ is not bounded on $[0, \frac{\pi}{2}]$, then it's not integrable.