MATH 125B, HW3, Solution

5.4.6 (a) Since $\lim_{x\to b^-}\frac{f(x)}{g(x)} = L$, this implies that for any $\epsilon > 0$, there exists $\delta > 0$, such that for all $b-\delta < x < b$, we have

$$L-\epsilon \le \frac{f(x)}{g(x)} \le L+\epsilon.$$

Since both f and g are non-negative, then

$$f(x) \le (L+\epsilon)g(x).$$

Because g is given to be improperly integrable, so is f.

(b) If $0 < L < \infty$, choose $\epsilon = \frac{L}{2}$. $\lim_{x\to b^-}\frac{f(x)}{g(x)} = L$ implies that there exists $\delta > 0$, such that for all $b-\delta < x < b$, we have

$$f(x) \ge \frac{L}{2}g(x).$$

Since f is non-negative and g is given to be not improper integrable, neither is f.

If $L = \infty$, for any M, there exists $\delta > 0$, such that for all $b-\delta < x < b$, we have

$$\frac{f(x)}{g(x)} > M.$$

Choose M=1. Then $f(x) > g(x)$.

5.4.7 (a) Assume that $\lim_{x\to\infty} f(x) = L$ and $L \neq 0$. Without loss of generality, suppose $L > 0$. This implies that for all $\epsilon > 0$, there exists $M > 0$, such that for all $x > M$, we have

$$L-\epsilon \le f(x) \le L+\epsilon.$$

Choose $\epsilon = \frac{L}{2}$, then there exists $M > 0$, for all $x > M$,

$$f(x) \ge \frac{L}{2}.$$

Then

$$\int_1^\infty f(x)dx = \int_1^M f(x)dx + \int_M^\infty f(x)dx \ge \int_1^M f(x) + \int_M^\infty \frac{L}{2}dx.$$

Since the RHS is not improper integrable, neither is f. We find a contradiction, therefore, $L = 0$.

(b) It's easy to show that $\lim_{x\to\infty} f(x)$ doesn't exist.

$$\int_0^b f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx + \cdots + \int_{\lfloor b\rfloor-1}^{\lfloor b\rfloor} f(x)dx + \int_{\lfloor b\rfloor}^b f(x)dx$$

$$\le 1 + \frac{1}{2} + \cdots \frac{1}{2^{\lfloor b\rfloor-1}} + \frac{1}{2^{\lfloor b\rfloor}} \le 2$$

where $\lfloor b\rfloor = \max\{m \in Z | m \le x\}$.

Since f is non-negative, $\int_0^b f(x)dx$ is non-decreasing and bounded, therefore the limit exists and is finite.

8.2.3 Find two parallel planes and choose two non-parallel lines on these planes.

8.2.4 (a)

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

(b)

$$T_2 = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$$

(c)

$$T_3 = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

8.2.6 It's sufficient to show that

(1) The equation is in the form of $mx + ny + pz = q$, for $m, n, p \neq 0$ at the same time, i.e. it's a plane in the space R^3;

(2) Check that the three points a, b, c satisfy the equaiotn, i.e. these three points lie on the plane.