5.1. Let $X$ be $\text{bin}(n, \theta)$ where $\theta$ is uniform on $(0,1)$.

If $\theta = u$, then

$$P(X=m | \theta = u) = \binom{n}{m} \theta^m (1-\theta)^{n-m}$$

No density function since $\theta$ is uniform and $1-\theta = 1$.

$$P(X=m) = \int_0^1 P(X=m | \theta = u) \, du = \sum \binom{n}{m} \theta^m (1-\theta)^{n-m} \, du$$

$$= \frac{\Gamma(m+1) \Gamma(n+1)}{\Gamma(n+m+2)}$$

This function of $\theta$, $\beta$ is called the "beta" function.

and you can find this formula in Analysis books (Rudin).

$$P(X=m) = \binom{n}{m} \frac{m! (n-m)!}{(m+n+1)!} \binom{n}{m} \frac{m! (n-m)!}{(n+m+1)!} = \frac{1}{n+1}$$

Hence $X$ is uniformly distributed on $\{0, 1, \ldots, n\}$. 