MAT 135A Homework 3

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Solutions of one selected problem

Homework 3 (4): Pairwise independent: We need to show that for all $1 \leq i < j \leq n$, $1 \leq k < l \leq n$, and $A_{ij} \neq A_{kl}$ we have

$$\mathbb{P}(A_{ij} \cap A_{kl}) = \mathbb{P}(A_{ij})\mathbb{P}(A_{kl}).$$

First of all, using the Bayes theorem

$$\begin{aligned}\mathbb{P}(A_{ij}) &= \mathbb{P}(i\text{th and } j\text{th roll produce the same facevalue}) \\ &= \sum_{k=1}^{6} \mathbb{P}(j\text{th roll produce the facevalue } k \mid i\text{th roll produce the facevalue } k\ )\mathbb{P}(i\text{th roll produce the facevalue } k).\end{aligned}$$

But the ith and jth trials are independent of each other. Therefore

$$\begin{aligned}&\mathbb{P}(j\text{th roll produce the facevalue } k \mid i\text{th roll produce the facevalue } k\ ) \\ =\ &\mathbb{P}(j\text{th roll produce the facevalue } k) = \frac{1}{6}.\end{aligned}$$

Consequently,

$$\mathbb{P}(A_{ij}) = \sum_{k=1}^{6} \frac{1}{6} \cdot \frac{1}{6} = \frac{6}{36} = \frac{1}{6}.$$

Similarly, $\mathbb{P}(A_{kl}) = \frac{1}{6}$.

$A_{ij} \cap A_{kl}$ denotes the event that ith, jth trial have the same facevalue and kth, lth trial have the same facevalue[1]. When we roll a die n times, the total number of outcomes is 6^n.

Case 1: Assume that the indices i, j, k, l are different from each other. Let us count the number of events when ith roll=jth roll=$\boxed{4}$ and kth roll=lth roll=$\boxed{1}$. Since the outcomes of ith, jth, kth, and lth rolls have been fixed and the rest of $(n-4)$ outcomes is not fixed, therefore this can happen in 6^{n-4} ways. The same is true if we demand ith roll=jth roll=$\boxed{3}$ and kth roll=lth roll=$\boxed{5}$. Since we can demand any two numbers (same or different) from $\{1, 2, 3, 4, 5, 6\}$ for the ith, jth roll and kth, lth roll, therefore the total number of outcomes favorable to the event $A_{ij} \cap A_{kl}$ is $6 \cdot 6 \cdot 6^{n-4} = 6^{n-2}$. Consequently $\mathbb{P}(A_{ij} \cap A_{kl}) = \frac{6^{n-2}}{6^n} = \frac{1}{36}$.

Case 2: Assume that there is only one index overlap between $\{i, j\}$ and $\{k, l\}$. For example, $i = k$.[2] In this case $A_{ij} \cap A_{kl}$ denotes the event that all of ith, jth, and lth rolls have the same facevalue (we have omitted k because $i = k$). In this case the number of outcomes favorable to $A_{ij} \cap A_{kl}$ is $6 \cdot 6^{n-3} = 6^{n-2}$ due to the similar reason as we have discussed in the previous case. Therefore $\mathbb{P}(A_{ij} \cap A_{kl}) = \frac{6^{n-2}}{6^n} = \frac{1}{36}$.

So in any case, $\mathbb{P}(A_{ij} \cap A_{kl}) = \mathbb{P}(A_{ij})\mathbb{P}(A_{kl})$.

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[1] Note that it is different from the event $E :=$'all of ith, jth, kth, and lth trail have the same facevalue'. For example, if ith, jth roll produce 4 and kth, lth roll produce 1, then it is a part the event $A_{ij} \cap A_{kl}$, but it is not a part of the event E.

[2] Note that we can not have both $i = k$ and $j = l$. Then it would violate the fact that $A_{ij} \neq A_{kl}$.

Remark: $P(A_{ij}) = \frac{1}{6}$ could also be proved using the technique described in Case 1 and Case 2.

Not independent: Consider the events A_{12}, A_{23}, A_{13}. If the events are independent then we should have $\mathbb{P}(A_{12} \cap A_{23} \cap A_{13}) = \mathbb{P}(A_{12})\mathbb{P}(A_{23})\mathbb{P}(A_{13})$. But we see that $A_{12} \cap A_{23} \cap A_{13} = A_{12} \cap A_{23}$ because both of them indicate the event that the first, second, and the third rolls produce the same number. Now using the technique discussed in the *Case 2* above, we have $\mathbb{P}(A_{12} \cap A_{23}) = \frac{1}{36}$ i.e., $\mathbb{P}(A_{12} \cap A_{23} \cap A_{13}) = \frac{1}{36}$. On the other hand $\mathbb{P}(A_{12})\mathbb{P}(A_{23})\mathbb{P}(A_{13}) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \neq \mathbb{P}(A_{12} \cap A_{23} \cap A_{13})$.