Solutions of one selected problem

**Homework 3 (4):** Pairwise independent: We need to show that for all $1 \leq i < j \leq n$, $1 \leq k < l \leq n$, and $A_{ij} \neq A_{kl}$ we have

$$P(A_{ij} \cap A_{kl}) = P(A_{ij})P(A_{kl}).$$

First of all, using the Bayes theorem

$$P(A_{ij}) = P(i\text{th and } j\text{th roll produce the same facevalue})$$

$$= \sum_{k=1}^{6} P(j\text{th roll produce the facevalue } k \mid i\text{th roll produce the facevalue } k) P(i\text{th roll produce the facevalue } k).$$

But the $i\text{th and } j\text{th trials are independent of each other. Therefore}$

$$P(j\text{th roll produce the facevalue } k \mid i\text{th roll produce the facevalue } k) = \frac{1}{6}.$$ 

Consequently,

$$P(A_{ij}) = \sum_{k=1}^{6} \frac{1}{6} \cdot \frac{1}{6} = \frac{6}{36} = \frac{1}{6}.$$ 

Similarly, $P(A_{kl}) = \frac{1}{6}$.

$A_{ij} \cap A_{kl}$ denotes the event that $i\text{th, } j\text{th trial have the same facevalue and } k\text{th, } l\text{th trial have the same facevalue}$. When we roll a die $n$ times, the total number of outcomes is $6^n$.

**Case 1:** Assume that the indices $i, j, k, l$ are different from each other. Let us count the number of events when $i\text{th roll}=j\text{th roll}=4$ and $k\text{th roll}=l\text{th roll}=1$. Since the outcomes of $i\text{th, } j\text{th, } k\text{th, and } l\text{th rolls have been fixed and the rest of } (n - 4) \text{ outcomes is not fixed, therefore this can happen in } 6^{n-4} \text{ ways. The same is true if we demand } i\text{th roll}=j\text{th roll}=3 \text{ and } k\text{th roll}=l\text{th roll}=5\text{.}$ Since we can demand any two numbers (same or different) from $\{1, 2, 3, 4, 5, 6\}$ for the $i\text{th, } j\text{th roll and } k\text{th, } l\text{th roll, therefore the total number of outcomes favorable to the event } A_{ij} \cap A_{kl} \text{ is } 6 \cdot 6^{n-4} = 6^{n-2}$. Consequently $P(A_{ij} \cap A_{kl}) = \frac{6^{n-2}}{6^n} = \frac{1}{36}$.

**Case 2:** Assume that there is only one index overlap between $\{i, j\}$ and $\{k, l\}$. For example, $i = k$. In this case $A_{ij} \cap A_{kl}$ denotes the event that all of $i\text{th, } j\text{th, and } l\text{th rolls have the same facevalue (we have omitted } k \text{ because } i = k\text{). In this case the number of outcomes favorable to } A_{ij} \cap A_{kl} \text{ is } 6 \cdot 6^{n-3} = 6^{n-2} \text{ due to the similar reason as we have discussed in the previous case. Therefore } P(A_{ij} \cap A_{kl}) = \frac{6^{n-2}}{6^n} = \frac{1}{36}$.

So in any case, $P(A_{ij} \cap A_{kl}) = P(A_{ij})P(A_{kl})$.

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1 Note that it is different from the event $E := \{\text{all of } i\text{th, } j\text{th, } k\text{th, and } l\text{th roll have the same facevalue}\}$. For example, if $i\text{th, } j\text{th roll produce 4 and } k\text{th, } l\text{th roll produce 1, then it is a part the event } A_{ij} \cap A_{kl}, \text{ but it is not a part of the event } E.$

2 Note that we can not have both $i = k$ and $j = l$. Then it would violate the fact that $A_{ij} \neq A_{kl}$. 
Remark: $P(A_{ij}) = \frac{1}{6}$ could also be proved using the technique described in Case 1 and Case 2.

Not independent: Consider the events $A_{12}, A_{23}, A_{13}$. If the events are independent then we should have $P(A_{12} \cap A_{23} \cap A_{13}) = P(A_{12})P(A_{23})P(A_{13})$. But we see that $A_{12} \cap A_{23} \cap A_{13} = A_{12} \cap A_{23}$ because both of them indicate the event that the first, second, and the third rolls produce the same number. Now using the technique discussed in the Case 2 above, we have $P(A_{12} \cap A_{23}) = \frac{1}{36}$ i.e., $P(A_{12} \cap A_{23} \cap A_{13}) = \frac{1}{36}$. On the other hand $P(A_{12})P(A_{23})P(A_{13}) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \neq P(A_{12} \cap A_{23} \cap A_{13})$. 

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