Solution of one selected problem

**Problem 1:** Let \( \#H \) and \( \#T \) denote the number of heads and tails respectively in \( n \) tosses of a fair coin. We are interested in the random variable \( X = |\#H - \#T| \).

First of all, we observe that if \( n \) is an even number then the lowest value of \( X \) is zero. Because we can have \( \frac{n}{2} \) heads and \( \frac{n}{2} \) tails. In general, let \( \#H - \#T = k \). Since \( \#H + \#T = n \), we have \( \#H = \frac{n+k}{2} \) and \( \#T = \frac{n-k}{2} \). But \( \#H \) (=the number of heads) must be an integer. Therefore \( \frac{n+k}{2} \) is an integer i.e., \( n+k \) is an even number. So if \( n \) is even, \( k \) is also even and if \( n \) is odd, \( k \) is also odd.

We also know that \( 0 \leq \#H \leq n \). Therefore \( 0 \leq \frac{n+k}{2} \leq n \) i.e., \( -n \leq k \leq n \). So if \( n \) is even then \( \#H - \#T = k = 0, 2, 4, \ldots, n \) and if \( n \) is odd then \( k = \pm 1, \pm 3, \pm 5, \ldots, n \). So \( |\#H - \#T| \in \{0, 2, 4, \ldots, n\} \) if \( n \) is even and \( |\#H - \#T| \in \{1, 3, 5, \ldots, n\} \) if \( n \) is odd.

When \( n \) is odd

\[
\mathbb{P}(X = k) = \mathbb{P}(\#H - \#T = k) + \mathbb{P}(\#T - \#H = k) = \mathbb{P}(\#H = (n+k)/2) + \mathbb{P}(\#H = (n-k)/2) = \left( \frac{n+k}{2n} \right) \frac{1}{2^n} + \left( \frac{n-k}{2n} \right) \frac{1}{2^n} = \left( \frac{n+k}{2n} \right) \frac{1}{2^{n-1}}, \quad k = 1, 3, 5, \ldots, n.
\]

When \( n \) is even,

\[
\mathbb{P}(X = 0) = \mathbb{P}(\#H = n/2) = \left( \frac{n}{2n} \right) \frac{1}{2^n},
\]

\[
\mathbb{P}(X = k) = \mathbb{P}(\#H - \#T = k) + \mathbb{P}(\#T - \#H = k) = \mathbb{P}(\#H = (n+k)/2) + \mathbb{P}(\#H = (n-k)/2) = \left( \frac{n+k}{2n} \right) \frac{1}{2^n} + \left( \frac{n-k}{2n} \right) \frac{1}{2^n} = \left( \frac{n+k}{2n} \right) \frac{1}{2^{n-1}}, \quad k = 2, 4, 6, \ldots, n.
\]

**Remark:** If the coin is unfair i.e., \( \mathbb{P}(H) = p \) where \( 0 \leq p \leq 1 \) but \( p \) is not necessarily \( \frac{1}{2} \), then

\[
\mathbb{P}(X = 0) = \mathbb{P}(\#H = n/2) = \left( \frac{n}{2n} \right) p^{n/2}(1-p)^{n/2}, \quad \text{if } n \text{ is even}
\]

\[
\mathbb{P}(X = k) = \left( \frac{n+k}{2n} \right) p^{(n+k)/2}(1-p)^{(n-k)/2} + \left( \frac{n-k}{2n} \right) p^{(n-k)/2}(1-p)^{(n+k)/2},
\]

where \( k \in \{2, 4, 6, \ldots, n\} \) if \( n \) is even and \( k \in \{1, 3, 5, \ldots, n\} \) if \( n \) is odd.