MAT 135A — Homework 4

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Solution of one selected problem

Problem 1: Let $\#H$ and $\#T$ denote the number of heads and tails respectively in n tosses of a fair coin. We are interested in the random variable $X = |\#H - \#T|$.

First of all, we observe that if n is an even number then the lowest value of X is zero. Because we can have $\frac{n}{2}$ heads and $\frac{n}{2}$ tails. In general, let $\#H - \#T = k$. Since $\#H + \#T = n$, we have $\#H = \frac{n+k}{2}$ and $\#T = \frac{n-k}{2}$. But $\#H$ (=the number of heads) must be an integer. Therefore $\frac{n+k}{2}$ is an integer i.e., $n+k$ is an even number. So if n is even, k is also even and if n is odd, k is also odd.

We also know that $0 \leq \#H \leq n$. Therefore $0 \leq \frac{n+k}{2} \leq n$ i.e., $-n \leq k \leq n$. So if n is even then $\#H - \#T = k = 0, \pm 2, \pm 4, \ldots, \pm n$ and if n is odd then $k = \pm 1, \pm 3, \pm 5, \ldots, \pm n$. So $|\#H - \#T| \in \{0, 2, 4, \ldots, n\}$ if n is even and $|\#H - \#T| \in \{1, 3, 5, \ldots, n\}$ if n is odd.

When n is odd

$$\begin{aligned}
\mathbb{P}(X = k) &= \mathbb{P}(\#H - \#T = k) + \mathbb{P}(\#T - \#H = k) \\
&= \mathbb{P}(\#H = (n+k)/2) + \mathbb{P}(\#H = (n-k)/2) \\
&= \binom{n}{\frac{n+k}{2}} \frac{1}{2^n} + \binom{n}{\frac{n-k}{2}} \frac{1}{2^n} = \binom{n}{\frac{n+k}{2}} \frac{1}{2^{n-1}}, \quad k = 1, 3, 5, \ldots, n.
\end{aligned}$$

When n is even,

$$\begin{aligned}
\mathbb{P}(X = 0) &= \mathbb{P}(\#H = n/2) = \binom{n}{\frac{n}{2}} \frac{1}{2^n} \\
\mathbb{P}(X = k) &= \mathbb{P}(\#H - \#T = k) + \mathbb{P}(\#T - \#H = k) \\
&= \mathbb{P}(\#H = (n+k)/2) + \mathbb{P}(\#H = (n-k)/2) \\
&= \binom{n}{\frac{n+k}{2}} \frac{1}{2^n} + \binom{n}{\frac{n-k}{2}} \frac{1}{2^n} = \binom{n}{\frac{n+k}{2}} \frac{1}{2^{n-1}}, \quad k = 2, 4, 6, \ldots, n.
\end{aligned}$$

Remark: If the coin is unfair i.e., $\mathbb{P}(H) = p$ where $0 \leq p \leq 1$ but p is not necessarily $\frac{1}{2}$, then

$$\begin{aligned}
\mathbb{P}(X = 0) &= \mathbb{P}(\#H = n/2) = \binom{n}{\frac{n}{2}} p^{n/2} (1-p)^{n/2}, \quad \text{if } n \text{ is even} \\
\mathbb{P}(X = k) &= \binom{n}{\frac{n+k}{2}} p^{(n+k)/2} (1-p)^{(n-k)/2} + \binom{n}{\frac{n-k}{2}} p^{(n-k)/2} (1-p)^{(n+k)/2},
\end{aligned}$$

where $k \in \{2, 4, 6, \ldots, n\}$ if n is even and $k \in \{1, 3, 5, \ldots, n\}$ if n is odd.