MAT 135A — Homework 7

March 11, 2015 — Indrajit Jana

Solution of one selected problem

Problem 7: Since $X_i \sim Exp(\lambda)$, the pdf of each X_i is given by

$$f_{X_i}(x_i) = \lambda e^{-\lambda x_i}, \quad 0 < x_i < \infty.$$

The probability distribution function of X_i is given by

$$\begin{aligned} F_{X_i}(x) &= \mathbb{P}(X_i \le x) \\ &= \int_0^x \lambda e^{-\lambda x_i}\, dx_i \\ &= 1 - e^{-\lambda x}, \quad x > 0. \end{aligned} \tag{1}$$

The probability distribution function of $H = \max\{X_1, \ldots, X_{10}\}$ is given by

$$\begin{aligned} F_H(x) &= \mathbb{P}(H \le x) \\ &= \mathbb{P}(X_1 \le x, X_2 \le x, \ldots, X_{10} \le x) \quad (\text{since } \max\{X_1, \ldots, X_{10}\} \le x \iff X_i \le x\ \forall i) \\ &= \prod_{i=1}^{10} \mathbb{P}(X_i \le x) \quad (\text{since } X_i \text{ s are independent of each other}) \\ &= \left(1 - e^{-\lambda x}\right)^{10} \quad (\text{using (1)}). \end{aligned}$$

Therefore the probability density function of H is given by

$$\begin{aligned} f_H(x) &= \frac{d}{dx} F_H(x) \\ &= 10\lambda e^{-\lambda x}(1 - e^{-\lambda x})^9, \quad x > 0, \end{aligned}$$

observing the face that $X_i > 0 \Rightarrow H > 0$.

Similarly we can compute the probability distribution function of $L = \min\{X_1, \ldots, X_{10}\}$

$$\begin{aligned} F_L(x) &= \mathbb{P}(L \le x) \\ &= 1 - \mathbb{P}(L > x) \\ &= 1 - \mathbb{P}(X_1 > x, \ldots, X_{10} > x) \quad (\text{since } \min\{X_1, \ldots, X_{10}\} > x \iff X_i > x\ \forall i) \\ &= 1 - \prod_{i=1}^{10} \mathbb{P}(X_i > x) \quad (\text{since } X_i \text{ s are independent of each other}) \\ &= 1 - e^{-10\lambda x} \quad (\text{using (1)}). \end{aligned}$$

Therefore the pdf of L is

$$\begin{aligned} f_L(x) &= \frac{d}{dx} F_L(x) \\ &= 10\lambda e^{-10\lambda x}, \quad x > 0. \end{aligned}$$