MAT 135A
Midterm 1 Solutions

Last Name (PRINT): Jana

First Name (PRINT): Indrajit

Student ID #: ____________

Section: All

Instructions:

1. Do not open your test until you are told to begin.
2. Use a pen to print your name in the spaces above.
3. No notes, books, calculators, smartphones, iPods, etc allowed.
4. Show all your work.
   Unsupported answers will receive NO CREDIT.
5. You are expected to do your own work.

#1	#2	#3	#4	#5	TOTAL

1. There are 30 mathematicians and 24 physicists attending a certain conference. Three of these 54 people are randomly chosen to take part in a panel

   discussion. What is the probability that at least one mathematician is chosen?

   Solution: Let A be the event that at least one mathematician is chosen. Then A^c is the event that no mathematician is chosen i.e., all three people are physicists. Since there are total 54 people, we can choose three people in $\binom{54}{3}$ different ways. Among those possibilities only $\binom{24}{3}$ cases will have all physicists, because there are total 24 physicists. Therefore

$$\mathbb{P}(A^c) = \frac{\binom{24}{3}}{\binom{54}{3}}.$$

   Consequently,

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{\binom{24}{3}}{\binom{54}{3}}.$$

2. You ask your neighbor to water a sickly plant while you are on vacation. Without water the plant will die with probability 0.8. With water it

   will die with probability 0.15. You are 90 percent certain the neighbor will remember to water the plant.

   (a) What is the probability that the plant will be alive when you return?

   (b) If it is dead what is the probability that your neighbor forgot to water it?

Solution: Let us define the following events.

$$\begin{aligned} A &:= \text{The plant is alive when I return,} \\ W &:= \text{My neighbor will remember to water it,} \\ X^c &:= \text{Complement of the event } X. \end{aligned}$$

According to the given information, we have

$$\mathbb{P}(W) = 0.9,\ \mathbb{P}(A^c|W^c) = 0.8,\ \mathbb{P}(A^c|W) = 0.15.$$

Therefore

$$\mathbb{P}(W^c) = 1 - 0.9 = 0.1,\ \mathbb{P}(A|W^c) = 1 - 0.8 = 0.2,\ \mathbb{P}(A|W) = 1 - 0.15 = 0.85.$$

(a) Using the Bayes theorem we obtain that the probability of the plant being alive is

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}(A|W)\mathbb{P}(W) + \mathbb{P}(A|W^c)\mathbb{P}(W^c) \\ &= (0.85 \times 0.9) + (0.2 \times 0.1) \\ &= 0.785 \end{aligned}$$

(b) According to our definition of the events, $W^c|A^c$ denotes the event that my neighbor forgot to water it given that the plant is dead upon my arrival. Using the conditional probability formula we obtain that

$$\begin{aligned} \mathbb{P}(W^c|A^c) &= \frac{\mathbb{P}(A^c|W^c)\mathbb{P}(W^c)}{\mathbb{P}(A^c)} \\ &= \frac{0.8 \times 0.1}{1 - 0.785} \quad \text{(from } \textit{part (a)} \text{ we have } \mathbb{P}(A) = 0.785) \\ &= \frac{0.08}{0.215} \\ &= \frac{16}{43}. \end{aligned}$$

3. A salesman has scheduled two appointments to sell encyclopedias. His first appointment will lead to a sale with probability .3, and his second will lead independently to a sale with probability .6. Any sale made is equally likely to be either for the deluxe model, which costs 1000 US Dollars, or the standard model, which costs 500 US Dollars.

   a) Determine the probability mass function of X, the total dollar value of all sales.
   b) Find $E(X)$.

Solution: Let us define the following events

$$\begin{aligned} A_i &:= \text{The salesman sales a book in his } i\text{th appointment,} \\ S_i &:= \text{He sells the standard model in his } i\text{th appointment,} \\ D_i &:= \text{He sells the deluxe model in his } i\text{th appointment,} \\ & \quad\; i = 1, 2. \end{aligned}$$

According to the given conditions

$$\mathbb{P}(A_1) = 0.3,\ \mathbb{P}(A_2) = 0.6,\ \mathbb{P}(S_i|A_i) = 0.5 = \mathbb{P}(D_i|A_i),\ i = 1, 2.$$

Therefore

$$\begin{aligned} \mathbb{P}(S_1) &= \mathbb{P}(S_1|A_1)\mathbb{P}(A_1) + \mathbb{P}(S_1|A_1^c)\mathbb{P}(A_1^c) = 0.5 \times 0.3 + 0 \times 0.7 = 0.15. \\ \text{Similarly, } \mathbb{P}(S_2) &= 0.5 \times 0.6 = 0.3,\ \mathbb{P}(D_1) = 0.15,\ \mathbb{P}(D_2) = 0.3. \end{aligned}$$

(a) The sales of the books in two different appointments can happen in these possible ways

$$\underbrace{A_1^c A_2^c}_{0}, \underbrace{S_1 A_2^c, A_1^c S_2}_{500}, \underbrace{A_1^c D_2, D_1 A_2^c, S_1 S_2}_{1000}, \underbrace{S_1 D_2, D_1 S_2}_{1500}, \underbrace{D_1 D_2}_{2000}.$$

The underbrace values denote the sale's values of those events. Since the sales in different appointments are independent of each other, the probability mass function of X is given by

$$\begin{aligned} \mathbb{P}(X = 0) &= \mathbb{P}(A_1^c A_2^c) = 0.7 \times 0.4 = 0.28 \\ \mathbb{P}(X = 500) &= \mathbb{P}(S_1 A_2^c) + \mathbb{P}(A_1^c S_2) = 0.15 \times 0.4 + 0.7 \times 0.3 = 0.27 \\ \mathbb{P}(X = 1000) &= \mathbb{P}(A_1^c D_2) + \mathbb{P}(D_1 A_2^c) + \mathbb{P}(S_1 S_2) = 0.7 \times 0.3 + 0.15 \times 0.4 + 0.15 \times 0.3 = 0.315 \\ \mathbb{P}(X = 1500) &= \mathbb{P}(S_1 D_2) + \mathbb{P}(D_1 S_2) = 0.15 \times 0.3 + 0.15 \times 0.3 = 0.09 \\ \mathbb{P}(X = 2000) &= \mathbb{P}(D_1 D_2) = 0.15 \times 0.3 = 0.045. \end{aligned}$$

(b)

$$\mathbb{E}[X] = (0 \times 0.28) + (500 \times 0.27) + (1000 \times 0.315) + (1500 \times 0.09) + (2000 \times 0.045) = 675$$

4. Suppose that there are n possible outcomes of a trial, with outcome i resulting with probability $p_i,\ i = 1, \ldots, n,\ \sum_i p_i = 1$. If two independent trials are observed, what is the probability that the result of the second trial is larger than the result of the first trial?

Solution: Let X_1, X_2 be the outcomes of the first and the second trial respectively. It is given that

$$\mathbb{P}(X_1 = i) = p_i = \mathbb{P}(X_2 = i), \quad i = 1, 2, \ldots, n.$$

Probability that the second trial is greater than the first trial is

$$\begin{aligned}
\mathbb{P}(X_2 > X_1) &= \sum_{i=1}^{n} \mathbb{P}(X_2 > X_1 | X_1 = i)\mathbb{P}(X_1 = i) \\
&= \sum_{i=1}^{n} \mathbb{P}(X_2 > i)\mathbb{P}(X_1 = i) \\
&= \sum_{i=1}^{n-1} \mathbb{P}(X_2 > i)\mathbb{P}(X_1 = i) + \mathbb{P}(X_2 > n)\mathbb{P}(X_1 = n) \\
&= \sum_{i=1}^{n-1} \left(\sum_{j=i+1}^{n} p_j \right) p_i + 0 \quad \text{(since } X_2 \text{ can't be bigger than } n\text{)} \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} p_j p_i .
\end{aligned}$$

Remark: We know that

$$\left(\sum_{i=1}^{n} p_i \right)^2 = \sum_{i=1}^{n} p_i^2 + 2 \sum_{1 \le i < j \le n} p_i p_j .$$

Therefore $\mathbb{P}(X_2 > X_1)$ can also be written as

$$\begin{aligned}
\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} p_j p_i &= \sum_{1 \le i < j \le n} p_i p_j \\
&= \frac{1}{2} \left[\left(\sum_{i=1}^{n} p_i \right)^2 - \sum_{i=1}^{n} p_i^2 \right] \\
&= \frac{1}{2} \left[1 - \sum_{i=1}^{n} p_i^2 \right],
\end{aligned}$$

since $\sum_{i=1}^{n} p_i = 1$.

5. Compute the probability that a hand of 13 cards contains (a) the ace and king of some suit; (b) all 4 of at least 1 of the 13 denominations.

Solution: (a) Let X_i be the event that the hand contains the Ace and King of the ith suit, $i = 1, 2, 3, 4$. 13 cards can be chosen in $\binom{52}{13}$ possible ways from the deck of 52 cards.

Now if we choose the Ace and King of the ith suit and the rest of 11 cards from any other cards, then the hand contains the Ace and King of the ith suit. This can be done in $\binom{50}{11}$ many ways. Therefore

$$\mathbb{P}(X_i) = \frac{\binom{50}{11}}{\binom{52}{13}}, \quad i = 1, 2, 3, 4.$$

Similarly,

$$\begin{aligned}
\mathbb{P}(X_i \cap X_j) &= \frac{\binom{48}{9}}{\binom{52}{13}}, \quad 1 \le i < j \le 4, \\
\mathbb{P}(X_i \cap X_j \cap X_k) &= \frac{\binom{46}{7}}{\binom{52}{13}}, \quad 1 \le i < j < k \le 4, \\
\mathbb{P}(X_1 \cap X_2 \cap X_3 \cap X_4) &= \frac{\binom{44}{5}}{\binom{52}{13}}.
\end{aligned}$$

The event $X_1 \cup X_2 \cup X_3 \cup X_4$ denotes the event that the hand contains the Ace and King of some suit. Using the inclusion exclusion formula we have

$$\begin{aligned}
\mathbb{P}(X_1 \cup X_2 \cup X_3 \cup X_4) &= \sum_{i=1}^{4} \mathbb{P}(X_i) - \sum_{1 \le i < j \le 4} \mathbb{P}(X_i \cap X_j) + \sum_{1 \le i < j < k \le 4} \mathbb{P}(X_i \cap X_j \cap X_k) \\
&\quad - \mathbb{P}(X_1 \cap X_2 \cap X_3 \cap X_4) \\
&= 4 \frac{\binom{50}{11}}{\binom{52}{13}} - \binom{4}{2} \frac{\binom{48}{9}}{\binom{52}{13}} + \binom{4}{3} \frac{\binom{46}{7}}{\binom{52}{13}} - \frac{\binom{44}{5}}{\binom{52}{13}}.
\end{aligned}$$

(b) Let Y_i be the event that the hand contains all 4 cards of the ith denomination. Clearly the event $\cup_{i=1}^{13} Y_i$ is the event of our interest. If we choose all four cards from the ith denomination and 9 other cards from the rest of 48 cards, then the event Y_i happens. These choices can be made in $\binom{48}{9}$ possible ways. Therefore

$$\mathbb{P}(Y_i) = \frac{\binom{48}{9}}{\binom{52}{13}}, \quad 1 \le i \le 13.$$

Similarly,

$$\mathbb{P}(Y_i \cap Y_j) = \frac{\binom{44}{5}}{\binom{52}{13}}, \quad 1 \le i < j \le 13, \quad \mathbb{P}(Y_i \cap Y_j \cap Y_k) = \frac{\binom{40}{1}}{\binom{52}{13}}, \quad 1 \le i < j < k \le 13.$$

Since we are choosing 13 cards, we can choose at most three complete denominations i.e., $\mathbb{P}(Y_{i_1} \cap Y_{i_2} \cap \ldots \cap Y_{i_l}) = 0$ if $l \ge 4$. So using the inclusion exclusion formula we have

$$\begin{aligned}
\mathbb{P}\left(\cup_{i=1}^{13} Y_i\right) &= \sum_{i=1}^{13} \mathbb{P}(Y_i) - \sum_{1 \le i < j \le 13} \mathbb{P}(Y_i \cap Y_j) + \sum_{1 \le i < j < k \le 13} \mathbb{P}(Y_i \cap Y_j \cap Y_k) \\
&= \binom{13}{1} \frac{\binom{48}{9}}{\binom{52}{13}} - \binom{13}{2} \frac{\binom{44}{5}}{\binom{52}{13}} + \binom{13}{3} \frac{\binom{40}{1}}{\binom{52}{13}}.
\end{aligned}$$