MAT 135A
Midterm 2 Solutions

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Student ID #: ________________

Section: All

Instructions:

1. Do not open your test until you are told to begin.
2. Use a pen to print your name in the spaces above.
3. No notes, books, calculators, smartphones, iPods, etc allowed.
4. Show all your work.
   Unsupported answers will receive NO CREDIT.
5. You are expected to do your own work.

#1	#2	#3	#4	TOTAL

1. Roll a fair die 4 times. Let X be the number of 6's obtained and Y be the number of 4's.

(a) Compute the joint probability density mass function of X and Y.

(b) Are X and Y independent?

Solution: (a) Since the die is rolled four times and each time there are six possible outcomes, therefore the total number of outcomes is 6^4. In each roll, there are three types of outcomes, namely 'six', 'four', 'neither six nor four'. i many 6's and j many 4's can occur in $\binom{4}{i}\binom{4-i}{j}\binom{4-i-j}{4-i-j}4^{4-i-j}$ many ways. Because, first of all pick i many rolls out of 4 rolls and assign the value 6 for those rolls, then we choose j many rolls from the rest and assign 4, and then assign any number other than 4 or 6 for the remaining $4-i-j$ rolls. Since it is a fair die, the joint probability mass function is given by

$$\begin{aligned}\mathbb{P}(X=i, Y=j) &= \frac{1}{6^4}\binom{4}{i}\binom{4-i}{j}\binom{4-i-j}{4-i-j}4^{4-i-j} \\ &= \frac{4!}{i!j!(4-i-j)!}\left(\frac{1}{6}\right)^i\left(\frac{1}{6}\right)^j\left(\frac{4}{6}\right)^{4-i-j}, \quad \text{when } i,j\in\mathbb{N}\ \&\ 0\le i+j\le 4.\end{aligned}$$

Remark: It is an application of the multinomial distribution.

(b) We observe that both X and Y are Binomial$(4, 1/6)$. Therefore

$$\begin{aligned}\mathbb{P}(X=i) &= \binom{4}{i}\left(\frac{1}{6}\right)^i\left(\frac{5}{6}\right)^{4-i}, \quad 0\le i\le 4, \\ \mathbb{P}(Y=j) &= \binom{4}{j}\left(\frac{1}{6}\right)^j\left(\frac{5}{6}\right)^{4-j}, \quad 0\le j\le 4,\end{aligned}$$

Clearly, $\mathbb{P}(X=i, Y=j)\neq\mathbb{P}(X=i)\mathbb{P}(Y=j)$. For example $\mathbb{P}(X=0,Y=0)=\left(\frac{4}{6}\right)^4\neq\left(\frac{5}{6}\right)^4\left(\frac{5}{6}\right)^4=\mathbb{P}(X=0)\mathbb{P}(Y=0)$. So, X and Y are not independent.

Grading rubric:

(a) (7 points)

- (+1) For computing the total number of cases.
- (+3) For computing the number of favorable cases.
- (+1) For writing the probability.
- (+2) For writing the range of the random variables.

(b) (3 points)

- (+1) For correct conclusion.
- (+2) For Explanation.

2. Assume that weekly sales of diesel fuel at a gas station are X tons, where X is a random variable with distribution function

$$f(x) = c(1-x)^4, \quad 0 < x < 1$$

(the density is zero outside the interval $(0,1)$).

(a) Compute EX.

(b) The current tank (of capacity 1) will be emptied and out for repairs next week; to save money, the station wants to rent a tank for the week with capacity just large enough that its supply will be exhausted with probability 0.01. What is the capacity of the tank?

Solution: Since f is a probability density function, $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= c\int_0^1 (1-x)^4\,dx \\ &= c\int_0^1 t^4\,dt \quad \text{(where } t = 1-x) \\ &= \frac{c}{5}.\end{aligned}$$

Therefore $\frac{c}{5} = 1$ i.e., $c = 5$.

(a)

$$\begin{aligned}\mathbb{E}[X] &= 5\int_0^1 x(1-x)^4\,dx \qquad (1)\\ &= 5\int_0^1 (1-t)t^4\,dt \quad \text{(where } t = 1-x) \\ &= 5\left[\frac{t^5}{5} - \frac{t^6}{6}\bigg|_0^1\right] \\ &= 5[1/5 - 1/6] = 1/6.\end{aligned}$$

(b) Suppose the capacity of the new tank is u. According to the given condition $\mathbb{P}(X \geq u) = 0.01$. In other words,

$$\begin{aligned}0.01 &= 5\int_u^1 (1-x)^4\,dx \qquad (2)\\ &= 5\int_0^{1-u} t^4\,dt \quad \text{(where } t = 1-x) \\ &= (1-u)^5.\end{aligned}$$

Solving the above equation, we have $u = 1 - (0.01)^{1/5}$.

Grading rubric:

(a) (5 points)

- (+2) For computing $c = 5$.
- (+2) For setting up the integral (1).
- (+1) For computation.

(b) (5 points)

- (+3) For setting up the integral (2)
- (+2) For computation.

3. Mr. Smith arrives at a location at a time uniformly distributed between 12:15 and 12:45, while Mrs. Smith independently arrives at the same location at a time uniformly distributed between 12 and 1 (all times p.m.).

1. Compute the probability that that the first person to arrive waits no longer than 5 minutes.
2. Compute the probability that Mr. Smith arrives first.

Solution: Let us define the following random variables.

$$\begin{aligned} X &:= \text{Mr. Smith's arrival time,} \\ Y &:= \text{Mrs. Smith's arrival time.} \end{aligned}$$

According to the given condition $X \sim U[15, 45]$, $Y \sim U[0, 60]$ and they are independent. So the joint *pdf* of (X, Y) is given by

$$f(x,y) = \begin{cases} \frac{1}{30} \cdot \frac{1}{60}, & 15 \le x \le 45, 0 \le y \le 60 \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} \frac{1}{1800}, & 15 \le x \le 45, 0 \le y \le 60 \\ 0 & \text{otherwise.} \end{cases}$$

(a) $|Y - X| \le 5$ denotes the event that the first person to arrive waits no longer than 5 minutes.

$$\begin{aligned} \mathbb{P}(|Y - X| \le 5) &= \mathbb{P}(-5 \le Y - X \le 5) \\ &= \mathbb{P}(X - 5 \le Y \le X + 5) \\ &= \int_{15}^{45} \int_{x-5}^{x+5} \frac{1}{1800}\, dydx \\ &= \int_{15}^{45} \frac{10}{1800}\, dx = \frac{30 \times 10}{1800} = \frac{1}{6}. \end{aligned}$$

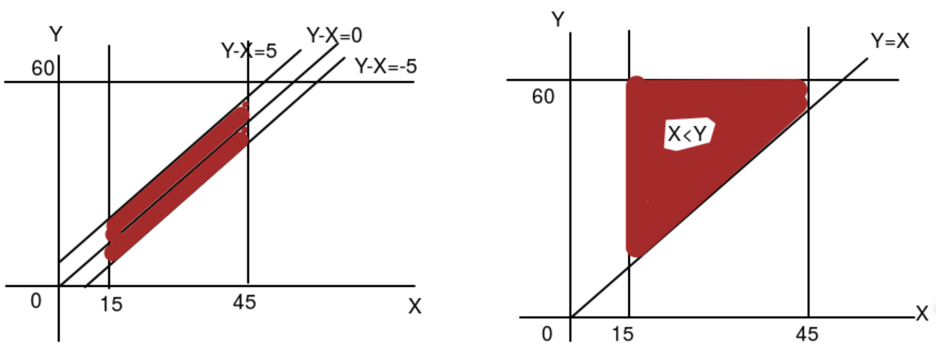

Figure 1: (a) $|X - Y| \le 5$ (b) $X < Y$

(b) $X \le Y$ denotes the event that Mr. Smith arrives first.

$$\begin{aligned} \mathbb{P}(X \le Y) &= \int_{15}^{45} \int_{x}^{60} \frac{1}{1800}\, dydx \\ &= \frac{1}{1800} \int_{15}^{45} (60 - x)\, dx \\ &= \frac{1}{1800} \left[60x - \frac{x^2}{2} \Big|_{15}^{45} \right] \\ &= \frac{60 \times 30 - (2025 - 225)/2}{1800} = \frac{900}{1800} = \frac{1}{2}. \end{aligned}$$

4. Let X be uniformly distributed on $[0, \pi/2]$.

(a) Find the density function of $Y = \sin X$.

(b) Find $VarY$.

Solution: (a) Since $X \sim U[0, \pi/2]$, the pdf of X is given by $f_X(x) = 2/\pi,\ 0 \le x \le \pi/2$. Notice that if $0 \le x \le \pi/2$, then $0 \le \sin x \le 1$. Also $\sin x$ is a monotonic increasing bujective function on $[0, \pi/2]$. The distribution function of Y is given by

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \le y) \\ &= \mathbb{P}(\sin X \le y) \\ &= \mathbb{P}(X \le \sin^{-1} y) \\ &= \int_{-\infty}^{\sin^{-1} y} f_X(x)\, dx \\ &= \frac{2}{\pi} \int_0^{\sin^{-1} y} dx \\ &= \frac{2 \sin^{-1} y}{\pi}. \end{aligned}$$

Therefore the pdf of Y is given by

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{2}{\pi \sqrt{1 - y^2}}, \quad 0 < y < 1.$$

(b)

$$\begin{aligned} \mathbb{E}[Y] &= \int_{-\infty}^{\infty} y f_Y(y)\, dy \\ &= \frac{2}{\pi} \int_0^1 \frac{y}{\sqrt{1 - y^2}}\, dy \\ &= -\frac{2}{\pi} \sqrt{1 - y^2}\Big|_0^1 \\ &= \frac{2}{\pi}. \end{aligned}$$

$$\begin{aligned} \mathbb{E}[Y^2] &= \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy \\ &= \frac{2}{\pi} \int_0^1 \frac{y^2}{\sqrt{1 - y^2}}\, dy \\ &= \frac{2}{\pi} \int_0^{\pi/2} \sin^2 t\, dt \quad (\text{where } y = \sin t) \\ &= \frac{1}{\pi} \int_0^{\pi/2} (1 - \cos 2t)\, dt \\ &= \frac{1}{\pi} \cdot \frac{\pi}{2} - 0 = \frac{1}{2}. \end{aligned}$$

Therefore $Var[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = 1/2 - 4/\pi^2$.