MAT 135A — Homework 1,2

February 6, 2015 — Indrajit Jana

Solutions of some selected problems

Homework 1 (3): (a) First of all, five men can be chosen in $\binom{15}{5}$ ways from the set of fifteen men. Then for each of such choices, five women can be chosen in $\binom{15}{5}$ ways from the set of fifteen women. Therefore from the set of fifteen couples, five men and five women can be chosen in $\binom{15}{5}\binom{15}{5}$ ways[1].

After choosing five men and five women, the first woman has five possible ways of pairing up with a man. Then for each of her choices the second woman has four possible ways of pairing up and so on. Therefore there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ possible ways of pairing up the five men and five women. So finally, there are $\binom{15}{5}\binom{15}{5}5!$ ways of pairing up five men and women from a set of fifteen couples.

(b) Let us first take out John Smith and Jane Smith from the set of fifteen couples and pair them. Then basically we need to choose four men and four women from the remaining set of fourteen couples and pair up those four men and four women. This can be done in $\binom{14}{4}\binom{14}{4}4!$ possible ways. The reason behind this is exactly same as the reason described in *part (a)*. Therefore the number of cases when Mr and Mrs. Smith dance together is $\binom{14}{4}\binom{14}{4}4!$. Therefore

$$\mathbb{P}(\text{Mr. and Mrs. Smith dance together}) = \frac{\binom{14}{4}\binom{14}{4}4!}{\binom{15}{5}\binom{15}{5}5!}.$$

Homework 2 (3): (a)Without replacement: There are total 19 balls and we are choosing 3 balls one by one without replacement. The first ball can be chosen in 19 different ways and for each of such choices the second ball can be chosen in 18 different ways and for each choices of the first and the second ball, the third ball can be chosen in 17 different ways. Therefore the total number of possible outcomes is $19 \cdot 18 \cdot 17$.

(1) There are three possibilities. All of the chosen three balls are either red, green, or blue. The first ball can be red in 5 different ways, for each of such choices the second ball can be red in 4 different ways, and then for each of such choices the third ball can be red in 3 different ways. Therefore all three can be red in $5 \cdot 4 \cdot 3$ different ways. Similarly all can be green in $6 \cdot 5 \cdot 4$ ways, and all can be blue in $8 \cdot 7 \cdot 6$ ways. Therefore

$$\mathbb{P}(\text{All balls are of the same color}) = \frac{5 \cdot 4 \cdot 3 + 6 \cdot 5 \cdot 4 + 8 \cdot 7 \cdot 6}{19 \cdot 18 \cdot 17}.$$

(2) In this case each ball is coming from a different color. Let us say the first, second, and the third balls are respectively red, green, and blue. Since there are 5 red, 6 green, and 8 blue balls, this can happen in $5 \cdot 6 \cdot 8$ ways. But the colors of the first, second, and the third ball may not necessarily be red, green, and blue. It could be blue, red, green too, and that can happen in $8 \cdot 5 \cdot 6$ ways. So we have to consider the all possible ordering of red, green, blue and number of such ordering is 3!. Therefore there are $5 \cdot 6 \cdot 8 \cdot 3!$ possible ways of getting three different colors.

$$\mathbb{P}(\text{All three have different colors}) = \frac{5 \cdot 6 \cdot 8 \cdot 3!}{19 \cdot 18 \cdot 17}$$

[1] 'for each of such choices' indicates that we have to take the product of 'number of choices for men' and the 'number of choices of women'

Remark: Notice that the first answer is exactly $= \frac{\binom{5}{3}+\binom{6}{3}+\binom{8}{3}}{\binom{19}{3}}$. You will get this answer if you choose all three balls at once (NOT one-by-one). The second answer is exactly $= \frac{5 \cdot 6 \cdot 8}{\binom{19}{3}}$. This answer can be obtained if you choose all three balls at once. So the answer is same for all at once and one-by-one.

(b) With replacement: * Since we are replacing the ball that we have chosen, every time we have 19 choices. Therefore the total number of outcome is 19^3. *

(1) Like the 'without replacement' case, we have three possibilities here. All are red, green, or blue. But since we are replacing the chosen ball, all red, all green, and all blue can happen in 5^3, 6^3, and 8^3 ways respectively. The reason is same as * . . .*. Therefore

$$\mathbb{P}(\text{All balls have the same color }) = \frac{5^3 + 6^3 + 8^3}{19^3}.$$

(2) For exactly the same reason as described in *part (a)(2)*, the number of ways we can draw balls of three different colors is $5 \cdot 6 \cdot 8 \cdot 3!$. Therefore

$$\mathbb{P}(\text{All three have different colors}) = \frac{5 \cdot 6 \cdot 8 \cdot 3!}{19^3}.$$

Remark: You can not choose three balls all at once with replacement. So no comment!