Math 135A **Midterm 2** **Winter 2014**

NAME(print in CAPITAL letters, first name first): Key

NAME(sign): ____________________

ID#: ____________________

Instructions: There are four problems. Make sure that you have all 4 problems. You must show all your work to receive full credit. Do not simplify complicated expressions unless instructed to do so.

Points received:

1 ________

2 ________

3 ________

4 ________

TOTAL ________

1. (25 points.) Suppose that X and Y are independent and uniform on $[0,1]$.

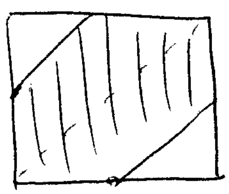
(a) Find $\mathbf{E}(X^3)$.

$$\int_0^1 x^3 f_X(x)\,dx$$

$$= \int_0^1 x^3 \cdot 1\,dx$$

$$= \left[\tfrac{1}{4}x^4\right]_0^1 = 1/4$$

(b) Find $\mathbf{P}(|X - Y| \le \frac{1}{2})$.

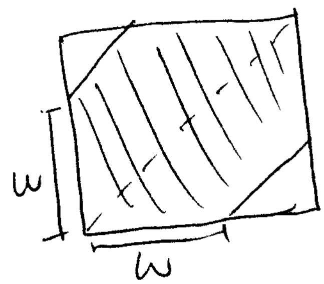


$\leftarrow$ area $= 3/4$

(c) Find the density f_W of the random variable $W = |X - Y|$.

$P(W \le w) =$

area (



) $= 1 - (1-w)^2$

$$f_W(w) = \frac{d}{dw} P(W \le w)$$

$$= 2(1-w) \quad \text{for } w \in [0,1]$$

2. (25 points.) A coin is tossed 1600 times.

(a) Find the probability that exactly 800 of the tosses land heads.

$$\binom{1600}{800}\left(\frac{1}{2}\right)^{1600}$$

(b) Using a relevant approximation, find the probability that there are less than 820 heads.

Let $S_n = \#$ heads $\qquad n=1600,\ p=1/2$

$$ES_n = np = 800$$
$$SD(S_n) = \sqrt{np(1-p)} = 20$$

$$P(S_n < 820) = P\left(\frac{S_n - 800}{20} < \frac{820-800}{20}\right)$$

$$\approx P(Z<1) = \Phi(1)$$

(c) Suppose that each toss lands in your beer with probability 1/1000, independently of all the other tosses. Using a relevant approximation, find the probability that exactly two tosses land in your beer.

approximately Poisson(λ)

for $\lambda = 1600 \cdot \frac{1}{1000} = 1.6$

$$\boxed{e^{-1.6}\ \frac{1.6^2}{2}}$$

3. (25 points.) Suppose that Y is exponential(1), X is uniform on $[0,1]$, and X and Y are independent.

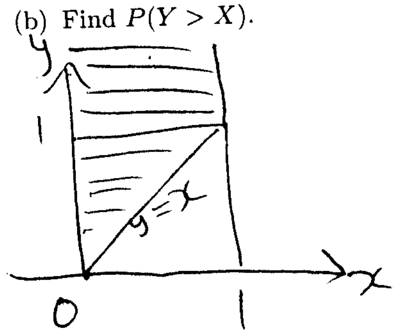
(a) Compute Var(X).

$$EX = \int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2}$$

$$EX^2 = \int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

$$\operatorname{Var} X = EX^2 - (EX)^2$$
$$= \tfrac{1}{3} - \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{12}$$

(b) Find $P(Y > X)$.

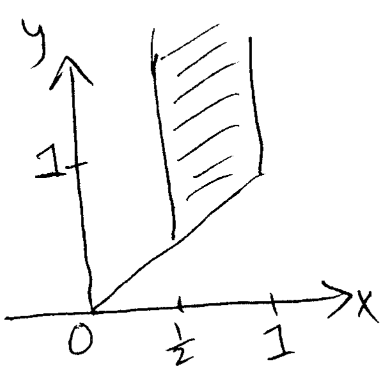


$$P(Y>X) = \int_0^1 \int_x^\infty e^{-y}\,dy\,dx$$
$$= \int_0^1 e^{-x}\,dx$$
$$= \boxed{1-e^{-1}}$$

$f(x,y) = e^{-y}$ for $y>0,\ 0 \le x \le 1$

(c) Find $P(Y > X \mid X \ge \frac{1}{2})$.

$$P(X \ge \tfrac{1}{2}) = \tfrac{1}{2}$$



$$P(Y>X, X \ge \tfrac{1}{2}) = \int_{1/2}^1 \int_x^\infty e^{-y}\,dy\,dx$$
$$= \int_{1/2}^1 e^{-x}\,dx$$
$$= e^{-1/2} - e^{-1}$$

Answer is $\dfrac{P(Y>X, X \ge \frac{1}{2})}{P(X \ge \frac{1}{2})} = 2(e^{-1/2} - e^{-1})$

4. (25 points.) A die is rolled twice. Let X be the number of 5's in the two rolls and let Y be the number of 6's.

(a) Write down the joint probability mass function of X and Y.

$$P(X=x, Y=y) = \binom{2}{x}\binom{2-i}{y}4^{2-x-y} \Big/ 6^2$$

y \ x	0	1	2
0	16/36	8/36	1/36
1	8/36	2/36	0
2	1/36	0	0

(b) Are X and Y independent? Explain.

No. $P(X=2) = 1/36$

$P(Y=1) = 10/36$

But $P(X=2, Y=1) = 0 \neq \frac{1}{36} \cdot \frac{10}{36}$

(c) Find $\mathbf{P}(X = 1 \mid X + Y = 2)$.

$$\frac{P(X=1, Y=1)}{P(X+Y=2)} = \frac{2/36}{\frac{1}{36} + \frac{2}{36} + \frac{1}{36}} = \frac{1}{2}$$