NAME(print in CAPITAL letters, first name first): Key

NAME(sign): __________________________

ID#: __________________________

Instructions: There are four problems. Make sure that you have all 4 problems. You must show all your work to receive full credit. Do not simplify complicated expressions unless instructed to do so.

Points received:
1 __________
2 __________
3 __________
4 __________
TOTAL ________
1. (25 points.) Suppose that $X$ and $Y$ are independent and uniform on $[0,1]$.

(a) Find $E(X^3)$.

$$\int_0^1 x^3 f_X(x) \, dx$$

$$= \int_0^1 x^3 \cdot 1 \, dx$$

$$= \left[ \frac{1}{4} x^4 \right]_0^1 = \frac{1}{4}$$

(b) Find $P(|X - Y| \leq \frac{1}{2})$.

area $= \frac{3}{4}$

(c) Find the density $f_W$ of the random variable $W = |X - Y|$.

$$P(W \leq w) =$$

area $\left( \frac{w}{w} \right) = 1 - (1-w)^2$

$$f_W(w) = \frac{d}{dw} P(W \leq w)$$

$$= 2(1-w)^2 \text{ for } w \in [0,1]$$
2. (25 points.) A coin is tossed 1600 times.

(a) Find the probability that exactly 800 of the tosses land heads.
\[
\binom{1600}{800} \left( \frac{1}{2} \right)^{1600}
\]

(b) Using a relevant approximation, find the probability that there are less than 820 heads.

Let \( S_n = \# \text{ heads} \) \( n = 1600 \), \( p = \frac{1}{2} \)

\[
E S_n = np = 800 \\
SD(S_n) = \sqrt{np(1-p)} = 20
\]

\[
P(S_n < 820) = P\left( \frac{S_n - 800}{20} < \frac{820 - 800}{20} \right)
\]

\[
\approx P(Z < 1) = \Phi(1)
\]

(c) Suppose that each toss lands in your beer with probability 1/1000, independently of all the other tosses. Using a relevant approximation, find the probability that exactly two tosses land in your beer.

approximately \( \text{Poisson}(\lambda) \)

for \( \lambda = 1600 \cdot \frac{1}{1000} = 1.6 \)

\[
e^{-1.6} \frac{1.6^2}{2}
\]
3. (25 points.) Suppose that Y is exponential(1), X is uniform on [0,1], and X and Y are independent.

(a) Compute \( \text{Var}(X) \).

\[
EX = \int_0^1 x \, dx = \left[ \frac{1}{2} x^2 \right]_0^1 = \frac{1}{2}
\]

\[
E[X^2] = \int_0^1 x^2 \, dx = \left[ \frac{1}{3} x^3 \right]_0^1 = \frac{1}{3}
\]

\[
\text{Var}(X) = E[X^2] - (EX)^2 = \frac{1}{3} - \left( \frac{1}{2} \right)^2 = \frac{1}{12}
\]

(b) Find \( P(Y > X) \).

\[
P(Y > X) = \int_0^\infty \int_x^\infty e^{-y} \, dy \, dx
\]

\[
= \int_x^\infty e^{-y} \, dy \Bigg|_0^\infty
\]

\[
= \left[ -e^{-y} \right]_0^\infty = 1 - e^{-x}
\]

(f) Find \( P(Y > X \mid X \geq \frac{1}{2}) \).

\[
P(X \geq \frac{1}{2}) = \frac{1}{2}
\]

\[
P(Y > X \mid X \geq \frac{1}{2}) = \int_{\frac{1}{2}}^{\infty} \int_x^\infty e^{-y} \, dy \, dx
\]

\[
= \int_{\frac{1}{2}}^{\infty} e^{-y} \, dy \bigg|_{\frac{1}{2}}^\infty
\]

\[
= e^{-\frac{1}{2}} - e^{-1}
\]

Answer is: \[ \frac{P(Y > X \mid X \geq \frac{1}{2})}{P(X \geq \frac{1}{2})} = 2(e^{-1/2} - e^{-1}) \]
4. (25 points.) A die is rolled twice. Let \( X \) be the number of 5's in the two rolls and let \( Y \) be the number of 6's.

(a) Write down the joint probability mass function of \( X \) and \( Y \).

\[
P(X=x, Y=y) = \binom{2}{x} \binom{2}{y} 1^{2-x-y} / 6^2
\]

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 6/36 & 8/36 & 1/36 \\
1 & 8/36 & 2/36 & 0 \\
2 & 1/36 & 0 & 0 \\
\end{array}
\]

(b) Are \( X \) and \( Y \) independent? Explain.

No. \( P(X=2) = 1/36 \), \( P(Y=1) = 10/36 \), but \( P(X=2, Y=1) = 0 \neq \frac{1}{36} \cdot \frac{10}{36} \).

(c) Find \( P(X = 1 | X + Y = 2) \).

\[
\frac{P(X=1, Y=1)}{P(X+Y=2)} = \frac{2/36}{1/36 + 2/36 + 1/36} = \frac{1}{2}
\]