MAT 135B
Midterm 2 Solutions

Last Name (PRINT): ________________________________

First Name (PRINT): ________________________________

Student ID #: ________________________________

Section: ________________________________

Instructions:

1. Do not open your test until you are told to begin.
2. Use a pen to print your name in the spaces above.
3. No notes, books, calculators, smartphones, iPods, etc allowed.
4. Show all your work.
   Unsupported answers will receive NO CREDIT.
5. You are expected to do your own work.

#1	#2	#3	#4	#5	TOTAL

1. For a branching process with $X_0 = 1$ and offspring distribution given by $p_0 = 1/3,\ p_1 = 1/3,\ p_2 = 1/3$, determine

   (a) the probability that the branching process dies by generation 3 but not by generation 1, and

   (b) the probability that the process ever dies out.

Solution: (a) The probability generating function of this branching process is given by

$$\begin{aligned}\phi(s) &= \sum p_k s^k \\ &= \frac{1}{3}(1 + s + s^2).\end{aligned}$$

Probability that the branching process dies out by generation 3 but not by generation 1 is

$$\begin{aligned}\phi(\phi(\phi(0))) - \phi(0) &= \phi(\phi(1/3)) - \frac{1}{3} \\ &= \phi(13/27) - \frac{1}{3} \\ &\approx 0.238\end{aligned}$$

(b) The probability π_0 that the process ever dies out is given by the smallest positive solution of the equation $\phi(s) = s$. Solving $\phi(s) = s$ boils down to solve the following quadratic equation.

$$\begin{aligned} & 1 + s + s^2 = 3s \\ \textit{i.e.,}\quad & s^2 - 2s + 1 = 0 \\ \textit{i.e.,}\quad & s = 1.\end{aligned}$$

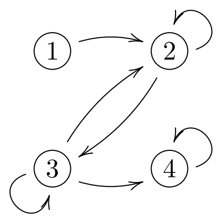
Therefore $\pi_0 = 1$ i.e., the process will eventually die out.

Remark: Check that the expected number of offsprings of each individual is $\mu = \frac{1}{3} + \frac{2}{3} = 1$. We know that if $\mu \leq 1$ then $\pi_0 = 1$.

2. Determine the transient and recurrent classes of the Markov chain with the following transition matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution: The following is a visual representation of the transition matrix.



From the above picture we see that $1 \to 2$, $2 \longleftrightarrow 3$, $3 \to 4$, and $4 \longleftrightarrow 4$. Therefore the classes are $\{1\}, \{2, 3\}$, and $\{4\}$. Since $1 \to 2$ and $3 \to 4$, the classes $\{1\}, \{2, 3\}$ not closed and therefore transient. The only closed class is $\{4\}$. So $\{4\}$ is the only recurrent class in the given Markov chain.

3. Prove or disprove that the simple symmetric random walk on Z^2 is recurrent.

Solution: Let S_n be the position of the simple symmetric random walk at the nth step. Since it is an irreducible Markov chain, if a single state is recurrent, all states are recurrent. Without loss of generality, assume $S_0 = (0,0)$ i.e., the random walk has started from the origin. Suppose that at nth step it has made k many steps along the X- axis and $n-k$ many steps along the Y-axis. Define

$$\begin{aligned} S_k^x &:= X \text{ coordinate of } S_n \\ S_{n-k}^y &:= Y \text{ coordinate of } S_n, \end{aligned}$$

Notice that S_k^x and S_{n-k}^y are independent simple symmetric random walks along the X-axis and Y-axis respectively. Let N be the number of horizontal steps then

$$\begin{aligned} P^{4n}_{(0,0),(0,0)} &= \mathbb{P}(\text{the random walk is at zero at the } 4n\text{th step}) \\ &= \sum_{k=0}^{2n} \mathbb{P}(S_{2k}^x = 0, S_{4n-2k}^y = 0)\mathbb{P}(N = 2k) \\ &= \sum_{k=0}^{2n} \mathbb{P}(S_{2k}^x = 0)\mathbb{P}(S_{4n-2k}^y = 0)\mathbb{P}(N = 2k) \\ &\geq \sum_{|k-n|\leq\sqrt{n}} \mathbb{P}(S_{2k}^x = 0)\mathbb{P}(S_{4n-2k}^y = 0)\mathbb{P}(N = 2k) \quad (\text{taking a particular choice } n - \sqrt{n} \leq k \leq n + \sqrt{n}). \end{aligned}$$

Since S_n^x is a simple symmetric random walk along the X-axis, $S_{2k}^x = 0$ if and only if we have k forward and k backward steps along the X-axis. So using the Binomial mass function and then the Stirling's formula $n! \sim \sqrt{2\pi n}(n/e)^n$ we have

$$\mathbb{P}(S_{2k}^x = 0) = \binom{2k}{k}\frac{1}{2^{2k}} = \frac{(2k)!}{(k!)^2}\frac{1}{2^{2k}} \sim \frac{\sqrt{4\pi k}(2k/e)^{2k}}{2\pi k(k/e)^{2k}}\frac{1}{2^{2k}} = \frac{1}{\sqrt{\pi k}}.$$

Similarly, $\mathbb{P}(S_{4n-2k}^y = 0) \sim 1/\sqrt{\pi(2n-k)}$. Therefore $\mathbb{P}(S_{2k}^x = 0)\mathbb{P}(S_{4n-2k}^y = 0) \sim 1/\pi\sqrt{k(2n-k)} \geq 1/\pi n$. Every step there is a equal chance to make a horizontal step or a vertical step. So $N \sim \textit{Binomial}(4n, 1/2)$. By Central Limit Theorem

$$\sum_{|k-n|\leq\sqrt{n}} \mathbb{P}(N = 2k)\frac{1}{2}\mathbb{P}(|N - 2n| \leq 2\sqrt{n}) \approx \frac{1}{2}\Phi(2),$$

where Φ is the CDF of the standard Normal distribution. So $P^{4n}_{(0,0),(0,0)} \geq \Phi(2)/2\pi n$. It is obvious that we need even number of steps to comeback to the origin. Therefore

$$\begin{aligned} \mathbb{E}[\text{number of returns to } (0,0)] &= \sum_{m=0}^{\infty} P^{2m}_{(0,0),(0,0)} \\ &\geq \sum_{n=0}^{\infty} P^{4n}_{(0,0),(0,0)} \quad (\text{taking a particular choice } m = 2n) \\ &\geq \sum_{n=0}^{\infty} \frac{\Phi(2)}{2\pi n} = \infty. \end{aligned}$$

So the simple symmetric random walk on $\mathbb{Z}^2$ is recurrent.

4. Let X_n be the size of the nth generation in an ordinary branching process with $X_0 = 1,\ E(X_1) = \mu$. Prove that

$$E(X_n X_m) = \mu^{n-m} E(X_m^2)$$

for $m \leq n$.

Solution: We know that if $X_0 = 1$ then $\mathbb{E}[X_l] = \mu^l$. Now if $X_0 = k$ then it can be treated as k many independent branching processes have been started together. If we add them up we get the total population. So $\mathbb{E}[X_l | X_0 = k] = \underbrace{\mu^l + \cdots + \mu^l}_{k \text{ many}} = k\mu^l$. Similarly, $\mathbb{E}[kX_l | X_0 = k] = k^2\mu^l$. Since $m \leq n$ and if it is given that $X_m = k$ then $\mathbb{E}[X_n X_m | X_m = k]$ can be thought of as the process has been started with k many individuals and we are looking at the $(n-m)$th generation. So $\mathbb{E}[X_n X_m | X_m = k] = \mathbb{E}[kX_{n-m}] = k^2\mu^{n-m}$. So

$$\mathbb{E}[X_n X_m] = \sum_{k=1}^{\infty} \mathbb{E}[X_n X_m | X_m = k]\mathbb{P}(X_m = k) = \mu^{n-m} \sum_{k=1}^{\infty} k^2 \mathbb{P}(X_m = k) = \mu^{n-m}\mathbb{E}[X_m^2].$$

<u>Alternative Solution</u>

Let W_i be the branching distribution of the ith individual in the $(n-1)$th generation. In other words

$$X_n = \sum_{i=1}^{X_{n-1}} W_i.$$

According to the given information, $\mathbb{E}[W_i] = \mu$ for all i. Also we know that the offspring distributions of each individual is independent of each other and it does not depend of the existing size of the population. In other words W_is are independent of $X_0, \ldots, X_m, \ldots, X_{n-1}$. Using conditional expectation we have

$$\begin{aligned}
\mathbb{E}[X_n X_m] &= \mathbb{E}\left[X_m \sum_{i=1}^{X_{n-1}} W_i\right] \\
&= \sum_{k=1}^{\infty} \mathbb{E}\left[X_m \sum_{i=1}^{X_{n-1}} W_i \middle| X_{n-1} = k\right] \mathbb{P}(X_{n-1} = k) \\
&= \sum_{k=1}^{\infty} \mathbb{E}\left[X_m \sum_{i=1}^{k} W_i \middle| X_{n-1} = k\right] \mathbb{P}(X_{n-1} = k) \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{k} \mathbb{E}\left[X_m W_i | X_{n-1} = k\right] \mathbb{P}(X_{n-1} = k) \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{k} \mathbb{E}[X_m | X_{n-1} = k]\mathbb{E}[W_i | X_{n-1} = k]\mathbb{P}(X_{n-1} = k) \quad \text{(since } W_i\text{s are independent of } X_m\text{)} \\
&= \mu \sum_{k=1}^{\infty} \sum_{i=1}^{k} \mathbb{E}[X_m | X_{n-1} = k]\mathbb{P}(X_{n-1} = k) \quad \text{(since } W_i\text{s are independent of } X_{n-1}\text{)} \\
&= \mu \sum_{k=1}^{\infty} \mathbb{E}[kX_m | X_{n-1} = k]\mathbb{P}(X_{n-1} = k) = \mu\mathbb{E}[X_{n-1}X_m].
\end{aligned}$$

Proceeding in this way, after $n-m$ steps we obtain $\mathbb{E}[X_n X_m] = \mu^{n-m}\mathbb{E}[X_m^2]$.

5. Three white and three black balls are distributed in two urns, with three balls per urn. The state of the system is the number of white balls in the first urn. At each step, we draw at random a ball from each of the two urns and exchange their places (the ball that was in the first urn is put into the second and vice versa).

   (a) Determine the transition matrix for this Markov chain.

   (b) Assume that initially all white balls are in the first urn. Determine the probability that this is also the case after 6 steps.

Solution: (a) Since the number of white balls in the first urn can be either $0, 1, 2$, or 3, the state space of the Markov chain is $\{0, 1, 2, 3\}$.

If the current state is 0 (i.e. the first urn has all three black balls and the second urn has all three white balls), then the next state must be 1. Because we are definitely going to pick a black ball from the first urn and a white ball from the second urn and exchange them. After the exchange is made, the first urn will contain only one white ball. Similarly, if the current state is 3 then the next state is definitely going to be 2.

For the other two cases, if the current state is $i \in \{1, 2\}$, then the next state can be $i - 1, i$ or $i + 1$. Probabilities of these transitions are discussed below.

Operation	Current State	Next State	Probability
white ball from the first urn and **black** ball from the second urn	**1**	**0**	1/9
	2	**1**	4/9
black ball from the first urn and **black** ball from the second urn	**1**	**1**	2/9
	2	**2**	2/9
white ball from the first urn and **white** ball from the second urn	**1**	**1**	2/9
	2	**2**	2/9
black ball from the first urn and **white** ball from the second urn	**1**	**2**	4/9
	2	**3**	1/9

Therefore the transition matrix is

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/9 & 4/9 & 4/9 & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(b) If initially all white balls are in the first urn then the Markov chain starts at the state 3. In other words the initial distribution is given by $\pi = [0\ \ 0\ \ 0\ \ 1]$. After six steps the distribution of the Markov chain is given by

$$[\mathbb{P}(X_6 = 0)\ \ \mathbb{P}(X_6 = 1)\ \ \mathbb{P}(X_6 = 2)\ \ \mathbb{P}(X_6 = 3)] = [0\ \ 0\ \ 0\ \ 1]P^6$$

Therefore $\mathbb{P}(X_6 = 3) = (P^6)_{44}$.