MAT 201B Homework 1

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THESE ARE NOT COMPLETE SOLUTIONS, AND SHOULD BE TREATED AS ROUGH SKETCHES

Problem 1.1: The proof given in the book seems more or less complete. We just need to show that the intersection of monotone classes containing an algebra $\mathcal{A}$ is a monotone class.

Let $Y = \cap \mathcal{S}$, where $\mathcal{S}$ are monotone classes containing $\mathcal{A}$. Let $A_i \in Y$, $i \in \mathbb{N}$ such that $A_1 \subset A_2 \subset A_3 \subset \cdots$. Since $A_i \in \mathcal{S}$ for all monotone classes containing $\mathcal{A}$, we have $\cup_i A_i \in \mathcal{S}$ for all $\mathcal{S}$ containing $\mathcal{A}$. Consequently, $\cup_i A_i \in Y$.

Similarly, Y is closed under the limit of decreasing sets.

Problem 1.2: Let $f : X \to \mathbb{R}$ be upper semi continuous, $x \in X$ and $\{x_n\}$ be a sequence converging to x. Then in particular, $f^{-1}(-\infty, t)$ is open for all $t > f(x)$. Note that $x \in f^{-1}(-\infty, t)$. Therefore there exists $\delta > 0$ and $N_\delta \in \mathbb{N}$ such that $x_n \in B(x; \delta) \subset f^{-1}(-\infty, t)$ for all $n \geq N_\delta$ i.e., $f(x_n) < t$ for all $n \geq \mathbb{N}_\delta$. Which implies $\limsup f(x_n) \leq t$, and this is true for all $t > f(x)$. Consequently, $\limsup f(x_n) \leq f(x)$.

Conversely, let "$\limsup f(x_n) \leq f(x)$ *for all* $x \in X$, *and for any* $x_n \to x$". Let $t \in \mathbb{R}$, and $x \in \overline{[f^{-1}(-\infty, t)]^c}$. Then $\exists$, $\{x_n\} \subset [f^{-1}(-\infty, t)]^c$ s.t. $x_n \to x$. By assumption, $t \leq \limsup f(x_n) \leq f(x)$, i.e., $x \in [f^{-1}(-\infty, t)]^c$. Which implies that $[f^{-1}(-\infty, t)]^c$ is closed.

Problem 1.3: Consider the collection $\mathcal{S} = \{A \in \mathcal{B}(\mathbb{R}) : f^{-1}(A) \in \Sigma\}$.

(i) Since f is Σ measurable, $(t, \infty) \in \mathcal{S}$, for all $t \in \mathbb{R}$

(ii) $\phi, \mathbb{R} \in \mathcal{S}$

(iii) Let $A \in \mathcal{S}$ i.e, $f^{-1}(A) \in \Sigma$, then $f^{-1}(A^c) = [f^{-1}(A)]^c \in \Sigma$ (since Σ is a $\sigma-$alg), $\Rightarrow A^c \in \mathcal{S}$.

(iv) Let $A_i \in \mathcal{S}, i \in \mathbb{N}$, then $f^{-1}(\cup_i A_i) = \cup_i f^{-1}(A_i) \in \Sigma$, $\Rightarrow \cup_i A_i \in \mathcal{S}$.

Therefore $\mathcal{S}$ is a σ-algebra containing the intervals (t, ∞). And we know that $\mathcal{B}(\mathbb{R})$ is the smallest of such σ-algebras. So $\mathcal{B}(\mathbb{R}) \subset \mathcal{S}$. We are done.

Problem 1.4: Let $\phi(z) = u(z) + iv(z)$. Enough to show both $u \circ f$ and $v \circ f$ are measurable. Since u is $\mathcal{B}(\mathbb{R}^2)$ measurable, $A_t := u^{-1}(t, \infty) \in \mathcal{B}(\mathbb{R}^2)$ for all $t \in \mathbb{R}$. Again f is Σ-measurable, so (by *Problem 1.3*) $f^{-1}(A_t) \in \Sigma$. Consequently, $(u \circ f)^{-1}(t, \infty) = f^{-1}(A_t) \in \Sigma$.

Similarly, it can be shown that $v \circ f$ is Σ measurable.

Problem 1.5: By the construction, $F_{f^j}(t)$ are a decreasing functions of t and $F_{f^j}(t) \uparrow F_f(t)$ for each $t \geq 0$.

We assume that f is integrable. Let $\epsilon > 0$, then there exists $M > 0$ such that $\int_M^\infty F_{f^j}(t)\, dt \leq \int_M^\infty F_f(t) < \epsilon$ (first inequality follows from $F_{f^j}(t) \leq F_f(t)\ \forall\, j \in \mathbb{N}, t \geq 0$). Let $0 = x_0 < x_1 < x_2 < \cdots < x_{2^k} = M$ be the partition of $[0, M]$ such that $x_i = \frac{iM}{2^k}$ and

$$U_k(F_f) - L_k(F_f) < \epsilon, \tag{1}$$

where $U_k(F_f) = \frac{M}{2^k} \sum_{i=0}^{2^k-1} F_f(x_i)$ and $L_k(F_f) = \frac{M}{2^k} \sum_{i=1}^{2^k} F_f(x_i)$ (since $F_f(t)$ is decreasing). Therefore $U_k(F_f) - L_k(F_f) = \frac{M}{2^k}(F_f(0) - F_f(M)) < \epsilon$.

Since $F_{f^j} \uparrow F_f$, we can find $j_\epsilon \in \mathbb{N}$, such that $0 \leq F_f(0) - F_{f^j}(0) < \epsilon$ and $0 \leq F_f(M) - F_{f^j}(M) < \epsilon$ for all $j \geq j_\epsilon$. Consequently for all $j \geq j_\epsilon$,

$$U_k(F_{f^j}) - L_k(F_{f^j}) = \frac{M}{2^k}(F_{f^j}(0) - F_{f^j}(M))$$

$$
\begin{aligned}
&\leq \frac{M}{2^k}\epsilon + \frac{M}{2^k}\left(F_f(0) - F_f(M)\right) \\
&< \epsilon + \frac{M}{2^k}\epsilon. \qquad (2)
\end{aligned}
$$

Using (1) and (2), we have

$$
\begin{aligned}
\left|\int_0^\infty F_{f^j}(t)\,dt - \int_0^\infty F_f(t)\,dt\right| &= \left|U_k(F_{f^j}) - \int_0^\infty F_{f^j}(t)\,dt + U_k(F_{f^j}) - U_k(F_f) + U_k(F_f) - \int_0^\infty F_f(t)\,dt\right| \\
&\leq \epsilon + \epsilon + \frac{M}{2^k}\epsilon + |U_k(F_{f^j}) - U_k(F_f)|
\end{aligned}
$$

Now we can take j large enough such that $0 \leq F_f(x_i) - F_{f^j}(x_i) < \epsilon/M$. Then we'll have $|U_k(F_{f^j}) - U_k(F_f)| < \epsilon$.