Problem 1.9: Notations: \( S_f(t) = \{ x \in \Omega : f(x) > t \} \), \( F_f(t) = \mu(S_f(t)) \).

(a) Notice \( S_{f+g}(t) \subset S_f(t/2) \cup S_g(t/2) \). Consequently,
\[
\int f + g \leq \int_0^\infty F_f(t/2) \ dt + \int_0^\infty F_g(t) \ dt = 2 \int f + 2 \int g.
\]

(b) Define
\[
f_N(x) = \begin{cases} \frac{(k - 1)/2^N}{2^N} & \text{if } (k - 1)/2^N \leq f(x) < k/2^N \text{ for } 1 \leq k \leq 2^N \\ 2^N & \text{if } f(x) \geq 2^N \end{cases}
\]

The rest follows from the MCT.

(c) Lemma 1. Let \( f(x) = \sum_{i=1}^n a_i \chi_{E_i}(x) \in L^1(\Omega) \) such that \( 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \), then \( \int_{\Omega} f(x) \ d\mu(x) = \sum_{i=1}^n a_i \mu(E_i) \).

Proof. Note that
\[
F_f(t) = \sum_{a_i > t} \mu(E_i).
\]

In particular, \( F_f(t) = 0 \) if \( t > a_n \). Therefore
\[
\int_{\Omega} f(x) \ d\mu(x) = \int_0^\infty F_f(t) \ dt = \sum_{i=1}^n \int_{a_{i-1}}^{a_i} F_f(t) \ dt + \int_{a_n}^\infty F_f(t) \ dt
\]
\[
= \sum_{i=1}^n \int_{a_{i-1}}^{a_i} \mu(E_i) \ dt + \int_{a_{n-1}}^{a_n} \mu(E_n) \ dt
\]
\[
= a_1 \sum_{i=1}^n \mu(E_i) + (a_2 - a_1) \sum_{i=2}^n \mu(E_i) + \cdots + (a_n - a_{n-1}) \mu(E_n)
\]
\[
= \sum_{i=1}^n a_i \mu(E_i).
\]

Now if \( f(x) = \sum_{i=1}^n a_i \chi_{E_i}(x) \in L^1(\Omega) \) and \( g(x) = \sum_{i=1}^m b_i \chi_{F_i}(x) \in L^1(\Omega) \), then \( f(x) + g(x) = \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j) \chi_{E_i \cap F_j}(x) \). From the lemma above,
\[
\int f + g = \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j) \mu(E_i \cap F_j)
\]
\[
= \sum_{i=1}^n \mu(E_i \cap F_j) + \sum_{j=1}^m b_j \mu(E_i \cap F_j) + \sum_{i=1}^n a_i \mu(E_i \cap F_j)
\]

1
Consequently, new space \( (\Omega, \mu) \)

(i) Let \( A \)

(ii) Let \( \ast \)

\[ \Omega = \{ x \in \Omega : f(x) \geq g(x) \} \]
\[ \Omega_g = \{ x \in \Omega : g(x) > f(x) \} . \]

Clearly, \( \Omega_f \cap \Omega_g = \phi \) and \( \Omega_f \cup \Omega_g = \Omega \). Then from the previous parts \( \int_{\Omega_f} (f-g) + \int_{\Omega_f} g = \int_{\Omega_f} f \) and \( \int_{\Omega_g} (g-f) + \int_{\Omega_g} f = \int_{\Omega_g} g \).

Consequently,
\[
\int_{\Omega} (f-g) = \int_{\Omega_f} (f-g) + \int_{\Omega_g} (f-g) \quad \text{(since } \Omega_f \cap \Omega_g = \phi \text{ and } \Omega_f \cup \Omega_g = \Omega) \\
= \int_{\Omega_f} f - \int_{\Omega_f} g + \int_{\Omega_g} g + \int_{\Omega_f} f \\
= \int_{\Omega} f - \int_{\Omega} g.
\]

(e) Follows directly.

**Problem 1.10:** [Completion of measure space] Let \((\Omega, \mathcal{F}, \mu)\) be the original measure space. Construct new space \((\Omega, \mathcal{F}^\ast, \mu^\ast)\) as

\[ \mathcal{F}^\ast = \{ A \cup N, A \setminus N : A \in \mathcal{F}, N \subset O, \mu(O) = 0 \}, \]
\[ \mu^\ast(A \cup N) = \mu(A) = \mu(A \setminus N) \quad \text{if } N \subset O \text{ such that } \mu(O) = 0. \]

\( \mathcal{F}^\ast \) is a sigma algebra

(i) Clearly, \( \Omega, \phi \in \mathcal{F}^\ast \).

(ii) Let \( A \cup N \in \mathcal{F}^\ast \), then \((A \cup N)^c = A^c \cap N^c = A \setminus N \in \mathcal{F}^\ast \). Similarly, if \( A \setminus N \in CF^\ast \) then \((A \setminus N)^c \in \mathcal{F}^\ast \).

(iii) Let \( A_i \cap N_i \in \mathcal{F}^\ast \), \( i \in \mathbb{N} \), then there exist \( O_i \in \mathcal{F} \) such that \( N_i \subset O_i \) and \( \mu(O_i) = 0 \). Clearly \( \mu(\bigcup_{i=1}^\infty O_i) \leq \sum_{i=1}^\infty \mu(O_i) = 0 \), and \( \bigcup_{i=1}^\infty N_i \subset \bigcup_{i=1}^\infty O_i \). Therefore \( \bigcup_{i=1}^\infty (A_i \cap N_i) = (\bigcup_{i=1}^\infty A_i) \cup (\bigcup_{i=1}^\infty N_i) \in \mathcal{F}^\ast \).

Similarly, if \( A_i \setminus M_i \in \mathcal{F}^\ast \), then \( \bigcup_{i=1}^\infty (A_i \setminus M_i) \in \mathcal{F}^\ast \) and \( [\bigcup_{i=1}^\infty (A_i \setminus M_i)] \cup [\bigcup_{i=1}^\infty (A_i \cup N_i)] \in \mathcal{F}^\ast \).

\( \mu^\ast \) is a measure: If \( A_i \cup N_i \in \mathcal{F}^\ast \), and \( A_i \cup N_i \) are disjoint from each other. Then
\[ \mu^\ast(\bigcup_{i=1}^\infty (A_i \cup N_i)) = \mu^\ast(\bigcup_{i=1}^\infty A_i) \cup (\bigcup_{i=1}^\infty N_i) = \mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i) = \sum_{i=1}^\infty \mu^\ast(A_i \cup N_i). \]
**Problem 1.12:** If \( f_n \geq 0 \) \( \forall n \), then the result follows from the MCT.

**Problem 1.13:** We assume that \( p < n \). We need the following two formula from [Wikipedia](https://en.wikipedia.org). The surface area \( S_{n-1}(r) \), and volume \( V_n(r) \) of a sphere of radius \( r \) in \( \mathbb{R}^n \) are given by

\[
S_{n-1}(r) = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1}, \\
V_n(r) = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} r^n.
\]

(i) Calculus method:

\[
\int_{\mathbb{R}^n} f(x) \, dx = \int_0^1 \int_{S_{n-1}(1)} r^{-p} r^{n-1} drdS \quad \text{(where } S_{n-1}(1) = \{ x \in \mathbb{R}^n : |x| = 1 \})
\]

\[
= \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{1}{n-p}
\]

(ii) Lebesgue’s definition: for \( 1 < a < \infty \),

\[
\mathcal{L}^n(\{ x : f(x) > a \}) = \mathcal{L}^n(\{ x : |x|^{-p} > a \})
\]

\[
= \mathcal{L}^n(\{ x : |x| < a^{-1/p} \})
\]

\[
= \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} a^{-n/p}.
\]

Therefore

\[
\int_{\mathbb{R}^n} f(x) \, dx = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} \int_1^\infty t^{-n/p} \, dt
\]

\[
= \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} \frac{n}{n-p}
\]

\[
= \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{1}{n-p}
\]

**Problem 1.17:** Let \( f(x) = \inf_{g \in \mathcal{F}} g(x) \). Therefore for any \( t \in \mathbb{R} \), \( \{ x : f(x) > t \} = \cap_{g \in \mathcal{F}} \{ x : g(x) > t \} \). Since \( g \in \mathcal{F} \) are continuous, all \( \{ x : g(x) > t \} \) are closed, and so is \( \{ x : f(x) > t \} \). Therefore \( \{ x : f(x) < t \} \) is open. Proved.

**Problem 1.18:** (a) If it holds for all \( a \in \mathbb{R} \), obviously it holds for all \( a \in \mathbb{Q} \).

Conversely, let \( b \in \mathbb{R} \setminus \mathbb{Q} \), and \( \{ a_n \} \subseteq \mathbb{Q} \) such that \( a_n \downarrow b \). Then \( \{ x : f(x) > b \} = \cup_{n=1}^\infty \{ x : f(x) > a_n \} \).

Thus \( \{ x : f(x) > b \} \), being a union of measurable sets, is measurable.

(b) Clearly, \( \cup_{b \in \mathbb{Q}} \{ x : f(x) > b \} \cap \{ x : g(x) > a - b \} \subseteq \{ x : f(x) + g(x) > a \} \). Conversely, let \( u \in \{ x : f(x) + g(x) > a \} \) and choose a \( b \in \mathbb{Q} \cap \{ a - g(u), f(u) \} \). Then \( f(u) > b \) and \( g(u) > a - b \). And thus \( \cup_{b \in \mathbb{Q}} \{ x : f(x) > b \} \cap \{ x : g(x) > a - b \} \supset \{ x : f(x) + g(x) > a \} \).
(c) In a similar way, \( \{ x : f(x)g(x) > a \} = \bigcup_{b \in \mathbb{Q} \setminus \{0\}} \{ x : f(x) > b \} \cup \{ x : g(x) > a/b \} \). Alternatively, define \( \Phi : \Omega \to \mathbb{R} \), and \( \Psi : \mathbb{R}^2 \to \mathbb{R} \) as \( \Phi(x) = (f(x), g(x)) \) and \( \Psi(u, v) = uv \). Then \( f(x)g(x) = (\Psi \circ \Phi)(x) \), being a composition of measurable functions, is measurable [Problem 1.3].

**Problem (HN) 6.1:** Let \( d = \inf_{x \in S} \|x\| \) and \( \{ x_n \} \subset S \) such that \( \|x_n\| \to d \). Using polarization identity, convexity of \( S \), and the definition of \( d \),
\[
\|x_n - x_m\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2 - 4\left\| \frac{1}{2}(x_n + x_m) \right\|^2 \\
\leq 2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2 \to 0 \quad \text{as} \quad m, n \to \infty.
\]
Thus \( \{ x_n \} \) is Cauchy. Completeness of \( \mathcal{H} \) and closedness of \( S \) implies \( x_n \to x \in S \) for some unique \( x \in S \).

**Problem (HN) 6.3:** (From 201A) First of all if \( S \subset T \), then from the definition of the orthogonal complement, we have \( T^\perp \subset S^\perp \). Consequently, \( \overline{T}^\perp \subset A^\perp \). Conversely, if \( u \in A^\perp \) and \( v \in \overline{T} \), then we can find a sequence \( \{ v_n \} \subset A \) such that \( \|v_n - v\| \to 0 \) and \( (u, v_n) = 0 \). As a result \( \|u\| = \|u - (u, v_n)v_n\| \leq \|u\|\|v - v_n\| \to 0 \). Which implies that \( u \in A^\perp \) is orthogonal to any vector \( v \in \overline{T} \) i.e., \( A^\perp \subset \overline{T}^\perp \) and thus \( A^\perp = \overline{T}^\perp \).

*Second part:* From the corollary 6.15, we can write \( \mathcal{H} = \overline{\mathcal{M}} + \overline{\mathcal{M}}^\perp \). Therefore \( \overline{\mathcal{M}} = \overline{\mathcal{M}}^{\perp\perp} = \mathcal{M}^{\perp\perp} \) (using previous part).

**Problem (HN) 6.8:** (From 201A) A more general question: When does \( \sum_{n=1}^{\infty} n^{-\alpha} x_n \) converges unconditionally but not absolutely?

*Answer:* Since \( \{ x_n \} \) forms an ONB, \( \sum_{n=1}^{\infty} a_n x_n \) converges absolutely iff \( \sum_{n=1}^{\infty} |a_n| < \infty \). In this case, the requirement boils down to \( \sum_{i=1}^{\infty} n^{-\alpha} < \infty \) i.e., \( \alpha > 1 \). Therefore the series \( \sum_{n=1}^{\infty} a_n x_n \) does not converge absolutely if \( \alpha \leq 1 \).

Since \( \{ x_n \} \) is an orthogonal set in the Hilbert space \( \mathcal{H} \), we have
\[
\left\| \sum_{n \in I} a_n x_n \right\|^2 = \sum_{n \in I} |a_n|^2 = \sum_{n \in I} n^{-2\alpha}, \quad I \subset \mathbb{N}.
\]
Therefore \( \sum_{n=1}^{\infty} n^{-\alpha} x_n \) converges unconditionally if \( \sum_{n=1}^{\infty} n^{-2\alpha} < \infty \) i.e., if \( \alpha > \frac{1}{2} \). So the series \( \sum_{n=1}^{\infty} n^{-\alpha} x_n \) converges unconditionally but not absolutely if \( \frac{1}{2} < \alpha \leq 1 \).

**Problem (HN) 6.5:** Try to prove that \( \bigoplus_{n=1}^{\infty} \mathcal{H}_n = S^\perp \), where \( S = \cap_{n=1}^{\infty} \mathcal{H}_n^\perp \). And use the fact that for any \( S \subset \mathcal{H} \), \( S^\perp \) is a closed linear subspace of \( \mathcal{H} \).

OR

Take \( \{ x^N \}_N \subset \bigoplus_{n=1}^{\infty} \mathcal{H}_n \) a Cauchy sequence. Then there exists \( N_0 \in \mathbb{N} \) such that \( \|x^M - x^N\| < \epsilon \) for all \( M, N \geq N_0 \).
Also notice that \( \{ x^N \}_N \subset \mathcal{H}_n \) are Cauchy sequences in \( \mathcal{H}_n \). Since \( \mathcal{H}_n \) are closed, \( x^N \to x_n \in \mathcal{H}_n \) as \( N \to \infty \).
Now
\[
\lim_{p \to \infty} \sum_{n=1}^{p} |x_n - x_n^{N_0}|^2 = \lim_{p \to \infty} \left( \lim_{N \to \infty} \sum_{n=1}^{p} |x_n^{N} - x_n^{N_0}|^2 \right) \\
\leq \lim_{p \to \infty} \epsilon^2 = \epsilon^2,
\]
which proves that \( x^N \to x = \{x_n\} \in \oplus_{n=1}^{\infty} \mathcal{H}_n \) and \( \|x\|_2 \leq \|x^N - x\|_2 + \|x^N_0\|_2 < \epsilon + \|x^N_0\|_2 < \infty. \)

**Problem (HN) 6.12:** *(From 201A)* Let \( f, g : \mathbb{R} \to \mathbb{R} \) have continuous derivatives up to the order \( n \), then the \( n \)th derivative of \( fg \) is given by the Leibniz rule
\[
(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x),
\]
where \( f^{(k)}(x) = \frac{d^k}{dx^k} f(x) \), and \( f^{(0)} = f \). We’ll use this rule several times in this problem.

(a)
\[
\begin{align*}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
&\vdots
\end{align*}
\]
From GM orthogonalizations of monomials
\[
\begin{align*}
f_0(x) &= 1 \\
f_1(x) &= x - \frac{\langle x, f_0 \rangle}{\langle f_0, f_0 \rangle} f_0(x) = x \\
f_2(x) &= x^2 - \frac{\langle x^2, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1(x) - \frac{\langle x^2, f_0 \rangle}{\langle f_0, f_0 \rangle} f_0(x) \\
&= x^2 - 1 + \frac{1}{3} \\
&\vdots
\end{align*}
\]
As we see, \( P_i \)'s are scalar multiples of \( f_i \)'s.

(e) Let \( h(x) = (x^2 - 1)^n = (x + 1)^n(x - 1)^n \). Differentiating \( h \) with respect to \( x \), we have
\[
(1 - x^2) h^{(1)}(x) = -2xh(x).
\]
Differentiating once again we have
\[
\begin{align*}
(1 - x^2) h^{(2)}(x) - 2xh^{(1)}(x) + 2nh(x) + 2nxh^{(1)}(x) &= 0 \\
i.e., \quad (1 - x^2) h^{(2)}(x) + 2x(n-1)h^{(1)}(x) + 2nh(x) &= 0
\end{align*}
\]
Differentiating the above equation with respect to $x$, and using the Leibniz rule we obtain

\[
\left[-2 \left(\frac{n}{2}\right) h^{(2n-2)}(x) - 2x \left(\frac{n}{1}\right) h^{(2n-1)}(x) + (1 - x^2)h^{(2n)}(x)\right] + 2(n - 1) \left[\left(\frac{n}{1}\right) h^{(1n-1)}(x) + xh^{(1n)}(x)\right] + 2nh^{(n)}(x) = 0
\]

\[i.e., \quad (1 - x^2)h^{(n+2)} - 2xh^{(n+1)}(x) + n(n + 1)h^{(n)}(x) = 0\]

\[i.e., \quad (1 - x^2)P_n^{(2)} - 2xP_n^{(1)}(x) + n(n + 1)P_n(x) = 0\]

\[i.e., \quad \frac{d}{dx} \left[ (1 - x^2) \frac{d}{dx} P_n(x) \right] = n(n + 1)P_n(x)\]

The above proves that $LP_n = \lambda_n P_n$, where $\lambda_n = n(n + 1)$.

(b) From part (c) we obtain $LP_m = \lambda_m P_m$ and $LP_n = \lambda_n P_n$. Therefore $(\lambda_m - \lambda_n)P_m P_n = P_nLP_m - P_mLP_n$, and thus

\[
(\lambda_m - \lambda_n) \int_{-1}^{1} P_m(x)P_n(x) \, dx = \int_{-1}^{1} P_m(x) \frac{d}{dx} \left[ (1 - x^2) \frac{d}{dx} P_n(x) \right] \, dx - \int_{-1}^{1} P_n(x) \frac{d}{dx} \left[ (1 - x^2) \frac{d}{dx} P_n(x) \right] \, dx
\]

\[= \left[ (1 - x^2)P_m(x)P_n(x) - (1 - x^2)P_n(x)P_m'(x) \right]_{-1}^{1}
\]

\[- \int_{-1}^{1} \left[ (1 - x^2)P_m(x)P_n'(x) - (1 - x^2)P_n'(x)P_m(x) \right] \, dx = 0\]

Since $\lambda_m \neq \lambda_n$ whenever $m \neq n$, the above proves that $\int_{-1}^{1} P_m(x)P_n(x) \, dx = 0$ whenever $m \neq n$. Therefore \(\{P_n\}_n\) are orthogonal to each other.

Using the Leibniz rule, we see that $P_n$ is a polynomial of degree $n$. Therefore span\{1, x, \ldots, x^n\} = span\{P_0, P_1, \ldots, P_n\} for all $n \geq 0$. So from the above, we can conclude that $P_n$ is orthogonal to span\{1, x, \ldots, x^{n-1}\}. Consequently, we can claim that the Legendre polynomials are obtained by the Gram-Schmidt orthogonalization of the monomials up to a normalization constant.

(c) Recall the function $h(x) = (x^2 - 1)^n = (x + 1)^n(x - 1)^n$ from the part (d). Using the Leibniz rule we obtain

\[
\frac{d^{n-1}}{dx^{n-1}} h(x) = n!(x + 1)^n(x - 1) + \sum_{k=1}^{n-2} \binom{n}{k} \frac{d^{k}}{dx^{k}} (x + 1)^n \left[ \frac{d^{n-1-k}}{dx^{n-1-k}} (x + 1)^n \right] + n!(x + 1)(x - 1)^n.
\]

From the above, have $\frac{d^{n-1}}{dx^{n-1}} h(x) \bigg|_{x=1} = 0 = \frac{d^{n-1}}{dx^{n-1}} h(x) \bigg|_{x=-1}$. Using the same method we can also prove that $\frac{d^{l}}{dx^{l}} h(x) \bigg|_{x=1} = 0 = \frac{d^{l}}{dx^{l}} h(x) \bigg|_{x=-1}, \quad 0 \leq l < n$.

Using the above observation and integration by parts, we have

\[
\int_{-1}^{1} \frac{d^{n}}{dx^{n}} (x^2 - 1)^n \frac{d^{n}}{dx^{n}} (x^2 - 1)^n \, dx = (-1)^n \int_{-1}^{1} (x^2 - 1)^n \frac{d^{2n}}{dx^{2n}} (x^2 - 1)^n \, dx
\]

\[= (-1)^n(2n)! \int_{-1}^{1} (x^2 - 1)^n \, dx \quad \text{(since } (x^2 - 1)^n = x^{2n} \text{ lower degree})
\]

\[= (-1)^n(2n)! \int_{0}^{\pi} \sin^{2n+1} u \, du, \quad x = \cos u
\]
\[
= 2(2n)! \int_0^{\pi/2} \sin^{2n+1} u \, du \\
= (2n)! \frac{\Gamma(n+1)\Gamma(1/2)}{\Gamma(n+3/2)} \\
= (2n)! \frac{n!2^{2n+1}}{(2n+1)(2n-1)(2n-3) \ldots 5 \cdot 3 \cdot 1} \\
= \frac{n!2^n n!2^{n+1}}{2n+1}.
\]

Using the above fact we obtain the result

\[
\int_{-1}^{1} (P_n(x))^2 \, dx = \frac{1}{(2^n n!)^2} \int_{-1}^{1} \frac{d^n}{dx^n} (x^2 - 1)^n \frac{d^n}{dx^n} (x^2 - 1) \, dx = \frac{2}{2n+1}.
\]

(d) We know that the continuous functions are dense in \( L^2[-1, 1] \) (w.r.t the \( \| \cdot \|_2 \) norm) and the polynomials are dense in the set of continuous functions (both w.r.t the \( \| \cdot \|_\infty \) and \( \| \cdot \|_2 \) norm). Therefore \( \text{span}\{1, x, \ldots\} = L^2[-1, 1] \). Then from part (a) we can say that the Legendre polynomials form an orthogonal basis of \( L^2[-1, 1] \).