MAT 201B — Homework 2

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The following solutions may not be complete, and should be treated as rough sketches

Problem 1.9: Notations: $S_f(t) = \{x \in \Omega : f(x) > t\}$, $F_f(t) = \mu(S_f(t))$.
(a) Notice $S_{f+g}(t) \subset S_f(t/2) \cup S_g(t/2)$. Consequently,

$$\int f + g \leq \int_0^\infty F_f(t/2)\, dt + \int_0^\infty F_g(t)\, dt = 2\int f + 2\int g.$$

(b) Define

$$f_N(x) = \begin{cases} (k-1)/2^N & \text{if } (k-1)/2^N \leq f(x) < k/2^N \text{ for } 1 \leq k \leq 2^{2N} \\ 2^N & \text{if } f(x) \geq 2^N. \end{cases}$$

The rest follows from the MCT.
(c)

Lemma 1. *Let $f(x) = \sum_{i=1}^n a_i \chi_{E_i}(x) \in L^1(\Omega)$ such that $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$, then $\int_\Omega f(x)\, d\mu(x) = \sum_{i=1}^n a_i \mu(E_i)$.*

Proof. Note that

$$F_f(t) = \sum_{a_i > t} \mu(E_i).$$

In particular, $F_f(t) = 0$ if $t > a_n$. Therefore

$$\begin{aligned}
\int_\Omega f(x)\, d\mu(x) &= \int_0^\infty F_f(t)\, dt \\
&= \sum_{i=1}^n \int_{a_{i-1}}^{a_n} F_f(t)\, dt + \int_{a_n}^\infty F_f(t)\, dt \\
&= \int_0^{a_1} \sum_{i=1}^n \mu(E_i)\, dt + \int_{a_1}^{a_2} \sum_{i=2}^n \mu(E_i)\, dt + \cdots + \int_{a_{n-1}}^{a_n} \mu(E_n)\, dt \\
&= a_1 \sum_{i=1}^n \mu(E_i) + (a_2 - a_1) \sum_{i=2}^n \mu(E_i) + \cdots + (a_n - a_{n-1})\mu(E_n) \\
&= \sum_{i=1}^n a_i \mu(E_i).
\end{aligned}$$

□

Now if $f(x) = \sum_{i=1}^n a_i \chi_{E_i}(x) \in L^1(\Omega)$ and $g(x) = \sum_{i=1}^m b_i \chi_{F_i}(x) \in L^1(\Omega)$, then $f(x) + g(x) = \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j)\chi_{E_i \cap F_j}(x)$. From the lemma above,

$$\begin{aligned}
\int f + g &= \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j)\mu(E_i \cap F_j) \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i \mu(E_i \cap F_j) + \sum_{i=1}^n \sum_{j=1}^m b_j \mu(E_i \cap F_j)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mu(E_i \cap F_j) + \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \mu(E_i \cap F_j) \\
&= \sum_{i=1}^{n} a_i \mu(E_i) + \sum_{j=1}^{m} b_j \mu(F_j) \\
&= \int f + \int g.
\end{aligned}
$$

(d) Combining the previous parts

$$
\begin{aligned}
\left|\int f+g - \int f - \int g\right| &= \left|\int f+g - \int f_N + g_N + \int f_N + g_N - \int f - \int g\right| \\
&= \left|\int f+g - \int f_N + g_N + \int f_N + \int g_N - \int f - \int g\right| \\
&\leq \left|\int f+g - \int f_N + g_N\right| + \left|\int f_N - \int f\right| + \left|\int g_N - \int g\right| \to 0
\end{aligned}
$$

Let $\Omega_f := \{x \in \Omega : f(x) \geq g(x)\}$ and $\Omega_g := \{x \in \Omega : g(x) > f(x)\}$. Clearly, $\Omega_f \cap \Omega_g = \phi$ and $\Omega_f \cup \Omega_g = \Omega$. Then from the previous parts $\int_{\Omega_f}(f-g) + \int_{\Omega_f} g = \int_{\Omega_f} f$ and $\int_{\Omega_g}(g-f) + \int_{\Omega_g} f = \int_{\Omega_g} g$. Consequently,

$$
\begin{aligned}
\int_{\Omega}(f-g) &= \int_{\Omega_f}(f-g) + \int_{\Omega_g}(f-g) \quad (\text{since } \Omega_f \cap \Omega_g = \phi \text{ and } \Omega_f \cup \Omega_g = \Omega) \\
&= \int_{\Omega_f} f - \int_{\Omega_f} g - \int_{\Omega_g} g + \int_{\Omega_f} f \\
&= \int_{\Omega} f - \int_{\Omega} g.
\end{aligned}
$$

(e) Follows directly.

Problem 1.10: *[Completion of measure space]* Let $(\Omega, \mathcal{F}, \mu)$ be the original measure space. Construct new space $(\Omega, \mathcal{F}^*, \mu^*)$ as

$$
\begin{aligned}
&\mathcal{F}^* = \{A \cup N, A \backslash N : A \in \mathcal{F}, N \subset O, \mu(O) = 0\}, \\
&\mu^*(A \cup N) = \mu(A) = \mu(A \backslash N) \quad \text{if } N \subset O \text{ such that } \mu(O) = 0.
\end{aligned}
$$

$\mathcal{F}^*$ is a sigma algebra

(i) Clearly $\Omega, \phi \in \mathcal{F}^*$.

(ii) Let $A \cup N \in \mathcal{F}^*$, then $(A \cup N)^c = A^c \cap N^c = A \backslash N \in \mathcal{F}^*$. Similarly, if $A \backslash N \in CF^*$ then $(A \backslash N)^c \in \mathcal{F}^*$.

(iii) Let $A_i \cup N_i \in \mathcal{F}^*, \quad i \in \mathbb{N}$, then there exist $O_i \in \mathcal{F}$ such that $N_i \subset O_i$ and $\mu(O_i) = 0$. Clearly $\mu(\cup_{i=1}^{\infty} O_i) \leq \sum_{i=1}^{\infty} \mu(O_i) = 0$, and $\cup_{i=1}^{\infty} N_i \subset \cup_{i=1}^{\infty} O_i$. Therefore $\cup_{i=1}^{\infty}(A_i \cup N_i) = (\cup_{i=1}^{\infty} A_i) \cup (\cup_{i=1}^{n} N_i) \in \mathcal{F}^*$.

Similarly, if $A_i \backslash M_i \in \mathcal{F}^*$, then $\cup_{i=1}^{\infty}(A_i \backslash M_i) \in \mathcal{F}^*$ and $[\cup_{i=1}^{\infty}(A_i \backslash M_i)] \cup [\cup_{i=1}^{\infty}(A_i \cup N_i)] \in \mathcal{F}^*$.

μ^* is a measure: If $A_i \cup N_i \in \mathcal{F}^*$, and $A_i \cup N_i$ are disjoint from each other. Then $\mu^*(\cup_{i=1}^{\infty}(A_i \cup N_i)) = \mu^*[(\cup_{i=1}^{\infty} A_i) \cup (\cup_{i=1}^{\infty} N_i)] = \mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \mu^*(A_i \cup N_i)$.

Problem 1.12: If $f_n \geq 0 \, \forall n$, then the result follows from the MCT.

Problem 1.13: We assume that $p < n$. We need the following two formula from Wikipedia. The surface area $S_{n-1}(r)$, and volume $V_n(r)$ of a sphere of radius r in $\mathbb{R}^n$ are given by

$$\begin{aligned} S_{n-1}(r) &= \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1} \\ V_n(r) &= \frac{\pi^{n/2}}{\Gamma(n/2+1)} r^n. \end{aligned}$$

(i) Calculus method:

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)\, dx &= \int_0^1 \int_{S_{n-1}(1)} r^{-p}\, r^{n-1} dr dS \ \ (\text{where } S_{n-1}(1) = \{x \in \mathbb{R}^n : |x| = 1\}) \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \cdot \frac{1}{n-p}. \end{aligned}$$

(ii) Lebesgue's definition: for $1 < a < \infty$,

$$\begin{aligned} \mathcal{L}^n(\{x : f(x) > a\}) &= \mathcal{L}^n(\{x : |x|^{-p} > a\}) \\ &= \mathcal{L}^n(\{x : |x| < a^{-1/p}\}) \\ &= \frac{\pi^{n/2}}{\Gamma(n/2+1)} a^{-n/p}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)\, dx &= \frac{\pi^{n/2}}{\Gamma(n/2+1)} \int_1^\infty t^{-n/p}\, dt \\ &= \frac{\pi^{n/2}}{\Gamma(n/2+1)} \frac{n}{n-p} \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \cdot \frac{1}{n-p}. \end{aligned}$$

Problem 1.17: Let $f(x) = \inf_{g \in \mathcal{F}} g(x)$. Therefore for any $t \in \mathbb{R}$, $\{x : f(x) \geq t\} = \cap_{g \in \mathcal{F}} \{x : g(x) \geq t\}$. Since $g \in \mathcal{F}$ are continuous, all $\{x : g(x) \geq t\}$ are closed, and so is $\{x : f(x) \geq t\}$. Therefore $\{x : f(x) < t\}$ is open. Proved.

Problem 1.18: (a) If it holds for all $a \in \mathbb{R}$, obviously it holds for all $a \in \mathbb{Q}$.

Conversely, let $b \in \mathbb{R} \backslash \mathbb{Q}$, and $\{a_n\} \subset \mathbb{Q}$ such that $a_n \downarrow b$. Then $\{x : f(x) > b\} = \cup_{n=1}^\infty \{x : f(x) > a_n\}$. Thus $\{x : f(x) > b\}$, being a union of measurable sets, is measurable.

(b) Clearly, $\cup_{b \in \mathbb{Q}} (\{x : f(x) > b\} \cap \{x : g(x) > a - b\}) \subset \{x : f(x) + g(x) > a\}$. Conversely, let $u \in \{x : f(x) + g(x) > a\}$ and choose a $b \in \mathbb{Q} \cap (a - g(u), f(u)]$. Then $f(u) > b$ and $g(u) > a - b$. And thus $\cup_{b \in \mathbb{Q}} (\{x : f(x) > b\} \cap \{x : g(x) > a - b\}) \supset \{x : f(x) + g(x) > a\}$.

(c) In a similar way, $\{x : f(x)g(x) > a\} = \cup_{b\in\mathbb{Q}\setminus\{0\}}(\{x : f(x) > b\} \cup \{x : g(x) > a/b\})$. Alternatively, define $\Phi : \Omega \to \mathbb{R}^2$, and $\Psi : \mathbb{R}^2 \to \mathbb{R}$ as $\Phi(x) = (f(x), g(x))$ and $\Psi(u, v) = uv$. Then $f(x)g(x) = (\Psi \circ \Phi)(x)$, being a composition of measurable functions, is measurable *[Problem 1.3]*.

Problem (HN) 6.1: Let $d = \inf_{x\in S} \|x\|$ and $\{x_n\} \subset S$ such that $\|x_n\| \to d$. Using polarization identity, convexity of S, and the definition of d,

$$\begin{aligned}\|x_n - x_m\|^2 &= 2\|x_m\|^2 + 2\|x_n\|^2 - 4\left\|\frac{1}{2}(x_n + x_m)\right\|^2\\ &\leq 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2 \to 0 \quad \text{as } m, n \to \infty.\end{aligned}$$

Thus $\{x_n\}$ is Cauchy. Completeness of $\mathcal{H}$ and closedness of S implies $x_n \to x \in S$ for some unique $x \in S$.

Problem (HN) 6.3: *(From 201A)* First of all if $S \subset T$, then from the definition of the orthogonal complement, we have $T^\perp \subset S^\perp$. Consequently, $\overline{A}^\perp \subset A^\perp$. Conversely, if $u \in A^\perp$ and $v \in \overline{A}$, then we can find a sequence $\{v_n\}_n \subset A$ such that $\|v_n - v\| \to 0$ and $\langle u, v_n\rangle = 0$. As a result $|\langle u, v\rangle| = |\langle u, v\rangle - \langle u, v_n\rangle| \leq \|u\|\|v - v_n\| \to 0$. Which implies that $u \in A^\perp$ is orthogonal to any vector $v \in \overline{A}$ i.e., $A^\perp \subset \overline{A}^\perp$ and thus $A^\perp = \overline{A}^\perp$.

Second part: From the corollary 6.15, we can write $\mathcal{H} = \overline{\mathcal{M}} \oplus \overline{\mathcal{M}}^\perp$. Therefore $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{\perp\perp} = \mathcal{M}^{\perp\perp}$ (using previous part).

Problem (HN) 6.8: *(From 201A)* A more general question: *When does $\sum_{n=1}^\infty n^{-\alpha} x_n$ converges unconditionally but not absolutely?*

Answer: Since $\{x_n\}$ forms an ONB, $\sum_{n=1}^\infty a_n x_n$ converges absolutely iff $\sum_{n=1}^\infty |a_n| < \infty$. In this case, the requirement boils down to $\sum_{i=1}^\infty n^{-\alpha} < \infty$ i.e., $\alpha > 1$. Therefore the series $\sum_{n=1}^\infty a_n x_n$ does not converge absolutely if $\alpha \leq 1$.

Since $\{x_n\}_n$ is an orthogonal set in the Hilbert space $\mathcal{H}$, we have

$$\left\|\sum_{n\in I} a_n x_n\right\|^2 = \sum_{n\in I} |a_n|^2 = \sum_{n\in I} n^{-2\alpha}, \quad I \subset \mathbb{N}.$$

Therefore $\sum_{n=1}^\infty n^{-\alpha} x_n$ converges unconditionally if $\sum_{n=1}^\infty n^{-2\alpha} < \infty$ i.e., if $\alpha > \frac{1}{2}$. So the series $\sum_{n=1}^\infty n^{-\alpha} x_n$ converges unconditionally but not absolutely if $\frac{1}{2} < \alpha \leq 1$.

Problem (HN) 6.5: Try to prove that $\oplus_{n=1}^\infty \mathcal{H}_n = S^\perp$, where $S = \cap_{n=1}^\infty \mathcal{H}_n^\perp$. And use the fact that for any $S \subset \mathcal{H}$, $S^\perp$ is a closed linear subspace of $\mathcal{H}$.

OR

Take $\{x^N\}_N \subset \oplus_{n=1}^\infty \mathcal{H}_n$ a Cauchy sequence. Then there exists $N_0 \in \mathbb{N}$ such that $\|x^M - x^N\|_2 < \epsilon$ for all $M, N \geq N_0$.

Also notice that $\{x_n^N\}_N \subset \mathcal{H}_n$ are Cauchy sequences in $\mathcal{H}_n$. Since $\mathcal{H}_n$ are closed, $x_n^N \to x_n \in \mathcal{H}_n$ as $N \to \infty$.

Now

$$
\begin{aligned}
\lim_{p\to\infty}\sum_{n=1}^{p}|x_n - x_n^{N_0}|^2 &= \lim_{p\to\infty}\left(\lim_{N\to\infty}\sum_{n=1}^{p}|x_n^N - x_n^{N_0}|^2\right)\\
&\le \lim_{p\to\infty}\epsilon^2 = \epsilon^2,
\end{aligned}
$$

which proves that $x^N \to x = \{x_n\} \in \oplus_{n=1}^{\infty}\mathcal{H}_n$ and $\|x\|_2 \le \|x^{N_0} - x\|_2 + \|x^{N_0}\|_2 < \epsilon + \|x^{N_0}\|_2 < \infty$.

Problem (HN) 6.12: *(From 201A)* Let $f, g : \mathbb{R} \to \mathbb{R}$ have continuous derivatives up to the order n, then the nth derivative of fg is given by the Leibniz rule

$$
(fg)^{(n)}(x) = \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(x) g^{(n-k)}(x),
$$

where $f^{(k)}(x) = \frac{d^k}{dx^k}f(x)$, and $f^{(0)} = f$. We'll use this rule several times in this problem.

(a)

$$
\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \frac{1}{2}(3x^2 - 1)\\
&\vdots
\end{aligned}
$$

From GM orthogonalizations of monomials

$$
\begin{aligned}
f_0(x) &= 1\\
f_1(x) &= x - \frac{\langle x, f_0\rangle}{\langle f_0, f_0\rangle} f_0(x) = x\\
f_2(x) &= x^2 - \frac{\langle x^2, f_1\rangle}{\langle f_1, f_1\rangle} f_1(x) - \frac{\langle x^2, f_0\rangle}{\langle f_0, f_0\rangle} f_0(x)\\
&= x^2 - \frac{1}{3}\\
&\vdots
\end{aligned}
$$

As we see, P_is are scalar multiples of f_is.

(e) Let $h(x) = (x^2 - 1)^n = (x+1)^n(x-1)^n$. Differentiating h with respect to x, we have

$$
(1 - x^2)h^{(1)}(x) = -2nxh(x).
$$

Differentiating once again we have

$$
\begin{aligned}
&(1 - x^2)h^{(2)}(x) - 2xh^{(1)}(x) + 2nh(x) + 2nxh^{(1)}(x) = 0\\
\textit{i.e.,}\quad &(1 - x^2)h^{(2)}(x) + 2x(n-1)h^{(1)}(x) + 2nh(x) = 0
\end{aligned}
$$

Differentiating the above equation with respect to x, and using the Leibniz rule we obtain

$$\begin{aligned}
& \left[-2\binom{n}{2}h^{(2+n-2)}(x) - 2x\binom{n}{1}h^{(2+n-1)}(x) + (1-x^2)h^{(2+n)}(x)\right] \\
& +2(n-1)\left[\binom{n}{1}h^{(1+n-1)}(x) + xh^{(1+n)}(x)\right] + 2nh^{(n)}(x) = 0 \\
\textit{i.e.,} \quad & (1-x^2)h^{(n+2)} - 2xh^{(n+1)}(x) + n(n+1)h^{(n)}(x) = 0 \\
\textit{i.e,} \quad & (1-x^2)P_n^{(2)} - 2xP_n^{(1)}(x) + n(n+1)P_n(x) = 0 \\
\textit{i.e.,} \quad & -\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right] = n(n+1)P_n(x).
\end{aligned}$$

The above proves that $LP_n = \lambda_n P_n$, where $\lambda_n = n(n+1)$.

(b) From *part (e)* we obtain $LP_m = \lambda_m P_m$ and $LP_n = \lambda_n P_n$. Therefore $(\lambda_m - \lambda_n)P_m P_n = P_n LP_m - P_m LP_n$, and thus

$$\begin{aligned}
(\lambda_m - \lambda_n)\int_{-1}^{1} P_m(x)P_n(x)\,dx &= \int_{-1}^{1} P_m(x)\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right]dx - \int_{-1}^{1} P_n(x)\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right]dx \\
&= \left[(1-x^2)P_m(x)P_n'(x) - (1-x^2)P_n(x)P_m'(x)\right]_{-1}^{1} \\
&\quad - \int_{-1}^{1}\left[(1-x^2)P_m'(x)P_n'(x) - (1-x^2)P_n'(x)P_m'(x)\right]dx \\
&= 0.
\end{aligned}$$

Since $\lambda_m \neq \lambda_n$ whenever $m \neq n$, the above proves that $\int_{-1}^{1} P_m(x)P_n(x) = 0$ whenever $m \neq n$. Therefore $\{P_n\}_n$ are orthogonal to each other.

Using the Leibniz rule, we see that P_n is a polynomial of degree n. Therefore $\text{span}\{1, x, \ldots, x^n\} = \text{span}\{P_0, P_1, \ldots, P_n\}$ for all $n \geq 0$. So from the above, we can conclude that P_n is orthogonal to $\text{span}\{1, x, \ldots, x^{n-1}\}$. Consequently, we can claim that the Legendre polynomials are obtained by the Gram-Schmidt orthogonalization of the monomials up to a normalization constant.

(c) Recall the function $h(x) = (x^2-1)^n = (x+1)^n(x-1)^n$ from the *part (d)*. Using the Leibniz rule we obtain

$$\frac{d^{n-1}}{dx^{n-1}}h(x) = n!(x+1)^n(x-1) + \sum_{k=1}^{n-2}\binom{n}{k}\left[\frac{d^k}{dx^k}(x+1)^n\right]\left[\frac{d^{n-1-k}}{dx^{n-1-k}}(x+1)^n\right] + n!(x+1)(x-1)^n.$$

From the above, have $\frac{d^{n-1}}{dx^{n-1}}h(x)\Big|_{x=1} = 0 = \frac{d^{n-1}}{dx^{n-1}}h(x)\Big|_{x=-1}$. Using the same method we can also prove that $\frac{d^l}{dx^l}h(x)\Big|_{x=1} = 0 = \frac{d^l}{dx^l}h(x)\Big|_{x=-1}$, $\forall\, 0 \leq l < n$.

Using the above observation and integration by parts, we have

$$\begin{aligned}
\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n\frac{d^n}{dx^n}(x^2-1)^n\,dx &= (-1)^n\int_{-1}^{1}(x^2-1)^n\frac{d^{2n}}{dx^{2n}}(x^2-1)^n\,dx \\
&= (-1)^n(2n)!\int_{-1}^{1}(x^2-1)^n\,dx \quad (\text{since } (x^2-1)^n = x^{2n} + \text{lower degree}) \\
&= (-1)^{2n}(2n)!\int_{0}^{\pi}\sin^{2n+1}u\,du, \quad x = \cos u
\end{aligned}$$

$$
\begin{aligned}
&= 2(2n)! \int_0^{\pi/2} \sin^{2n+1} u \; du \\
&= (2n)! \frac{\Gamma(n+1)\Gamma(1/2)}{\Gamma(n+3/2)} \\
&= (2n)! \frac{n! 2^{n+1}}{(2n+1)(2n-1)(2n-3)\ldots 5 \cdot 3 \cdot 1} \\
&= \frac{n! 2^n n! 2^{n+1}}{2n+1}.
\end{aligned}
$$

Using the above fact we obtain the result

$$
\int_{-1}^{n} (P_n(x))^2 \; dx = \frac{1}{(2^n n!)^2} \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^n}{dx^n}(x^2-1)^n \; dx = \frac{2}{2n+1}.
$$

(d) We know that the continuous functions are dense in $L^2[-1,1]$ (w.r.to the $\|\cdot\|_2$ norm) and the polynomials are dense in the set of continuous functions (both w.r.to the $\|\cdot\|_\infty$ and $\|\cdot\|_2$ norm). Therefore $\overline{\text{span}\{1, x, \ldots\}} = L^2[-1,1]$. Then from *part (a)* we can say that the Legendre polynomials form an orthogonal basis of $L^2[-1,1]$.