The following solutions may not be complete, and should be treated as rough sketches

Problem 7.1: (a)

\[
\frac{1}{c_n} = 2 \int_0^\pi 2^n \cos^{2n} \frac{x}{2} \, dx = 4 \int_0^{\pi/2} 2^n \cos^{2n} y \, dy = \frac{2^{n+1} \Gamma(1/2) \Gamma((2n+1)/2)}{\Gamma(n+1)} \sim 2^{2n} \sqrt{n} \quad \text{as } n \to \infty, \quad \text{(by Stirling’s approximation)}.
\]

Note that for any $0 < \delta < \pi$ if $x \in [-\pi, \delta] \cup [\delta, \pi]$, then we have $(1 + \cos x) \leq 1 + \cos \delta < 2$. Therefore $\frac{\sqrt{\pi}}{2\pi} (1 + \cos \delta)^n \to 0$ as $n \to \infty$ i.e. $\phi_n(x) = c_n (1 + \cos x)^n \to 0$ for $\delta \leq |x| \leq \pi$. Also $|\phi_n(x)1_{\delta \leq |x| \leq \pi}| \leq 1000 \frac{1 + \cos \delta}{\sqrt{2}} \in L^1[\delta, \pi]$. So by DCT we have $\int_{\delta \leq |x| \leq \pi} \phi_n(x) \, dx \to 0$.

Alternatively

\[
c_n \int_{\delta \leq |x| \leq \pi} (1 + \cos x)^n \, dx = \frac{\int_{\delta \leq |x| \leq \pi} (1 + \cos x)^n \, dx}{\int_{-\pi}^{\pi} (1 + \cos x)^n \, dx} = \frac{\int_{\delta \leq |x| \leq \pi} (1 + \cos \delta)^n \, dx}{\int_{-\pi}^{\pi} (1 + \cos x)^n \, dx} \leq \frac{2(\pi - \delta)(1 + \cos \delta)^n}{\delta(1 + \cos(\delta/2))^n} \to 0.
\]

(b) Continuous functions on $\mathbb{T}$ are periodic continuous, and therefore can be uniformly approximated by the trigonometric polynomials. As a consequence (of uniform convergence) we have $L^2$ convergence.

I think the actual question would be prove that $\mathcal{P}$ is dense in the space of continuous functions on $[0, 2\pi]$ with the $L^2$-norm.

Let $f \in C([0, 2\pi])$. Since $f$ is continuous on a compact interval, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [0, 2\pi]$. Let us take $\epsilon > 0$ and define $\delta = \epsilon^2/M^2$. Now define $\tilde{f}(x) = f(x)1_{(\delta, 2\pi-\delta)}(x) +$
\[
\frac{f(\delta)}{\delta} x \mathbf{1}_{[0, \delta]}(x) + \frac{f(2\pi - \delta)}{\delta} (2\pi - x) \mathbf{1}_{[2\pi - \delta, 2\pi]}(x). \text{ (i.e. } \hat{f} = f \text{ on } [\delta, 2\pi - \delta] \text{ and then linearly join to 0 after that).}
\]
Notice that \(\|f - \hat{f}\|_2 < 2\epsilon\).

Also \(\hat{f}(0) = 0 = \hat{f}(2\pi)\), i.e., it is periodic continuous on \(T\) and hence can be uniformly approximated by elements in \(P\). Let \(\phi \in P\) such that \(\|\phi - \hat{f}\|_\infty < \epsilon\). Then \(\|\phi - f\|_2 \leq \|\phi - \hat{f}\|_2 + \|\hat{f} - f\|_2 < \sqrt{2\pi} \epsilon + 2\epsilon\). Hence \(P\) is dense in \(C([0, 2\pi])\).

(c) If \(f(0) \neq f(2\pi)\), then it is discontinuous as a function on \(T\). And hence can not be uniformly approximated by continuous functions on \(T\).

**Problem 7.2:**

(a) Note that
\[
\sum_{n=-N}^{N} e^{inx} = \frac{1 - e^{i(2N+1)x}}{1 - e^{ix}}
\]
\[
= \frac{1 - \cos(2N+1)x - i\sin(2N+1)x}{1 - \cos x - i\sin x}
\]
\[
= \frac{2\sin(N+1/2)x[x\sin(N+1/2)x - i\cos(N+1/2)x]}{2\sin(x/2)[\sin(x/2) - i\cos(x/2)]}
\]
\[
= 2\pi D_n(x) e^{-iNx} \frac{-i\exp\{i(N+1/2)x\} - \exp\{-i(N+1/2)x\}}{-i\exp\{ix/2\}}
\]
\[
= 2\pi D_n(x).
\]

Therefore
\[
S_N = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \hat{f}_n e^{inx}
\]
\[
= \frac{1}{2\pi} \sum_{n=-N}^{N} \int_T f(y) e^{in(x-y)} \, dy
\]
\[
= (D_N \ast f)(x).
\]

(b) Note that
\[
\sum_{n=0}^{N} \sin(n + 1/2)x = \frac{1}{2i} \sum_{n=0}^{N} \exp\{i(n + 1/2)x\} - \exp\{-i(n + 1/2)x\}
\]
\[
= \frac{1}{2i} \left[ \frac{1 - \exp\{i(N+1)x\}}{\exp\{-ix/2\} - \exp\{ix/2\}} + \frac{1 - \exp\{-i(N+1)x\}}{\exp\{-ix/2\} - \exp\{ix/2\}} \right]
\]
\[
= \frac{1}{2i} \left[ \frac{4\sin^2(N+1/2)x/2}{2\sin(x/2)} \right]
\]
\[
= \frac{\sin^2(N+1/2)x/2}{\sin(x/2)}.
\]

Therefore
\[
T_N = \frac{1}{N+1} \left( \sum_{n=0}^{N} D_n \right) \ast f = F_N \ast f,
\]
where
\[
F_N(x) = \frac{1}{2\pi(N+1)} \left( \frac{\sin(N+1)x/2}{\sin(x/2)} \right)^2.
\]

(c) \(D_N\) could be negative (e.g. \(D_1(\pi) = -1/2\pi\)), and thus can not be approximate identities.
On the other hand,
(i) \( F_N(x) \geq 0. \)

(ii) Notice from the description of \( T_N \) and \( S_N \)

\[
\int_{-\pi}^{\pi} F_N(x) \, dx = \frac{1}{2\pi(N+1)} \sum_{n=-N}^{N} (N+1-n) \int_{-\pi}^{\pi} e^{inx} \, dx = 1.
\]

(iii) For any \( \delta > 0, F_N(x) \leq \frac{1}{2\pi(N+1) \sin^2(\delta/2)} \) for all \( -\pi \leq |x| \leq \pi \). It follows that \( \int_{\delta \leq |x| \leq \pi} F_N(x) \, dx \to 0. \)

So \( \{F_N\} \) are approximation to identity.

From theorem 7.2, \( T_N \to f \) uniformly. However \( S_N \) may not converge uniformly.

**Problem 7.3:** We know that \( \{e^{inx} : n \in \mathbb{Z}\} \) forms an orthogonal basis of \( L^2[-\pi, \pi] \). Let \( f \in L^2[0, \pi] \).

(i) Extend \( f \) to an odd function \( F \in L^2[-\pi, \pi] \). Say \( F(x) = f(x)1_{[0,\pi]}(x) - f(-x)1_{[-\pi,0]}(x) \). Notice that \( \hat{F}_n = -\hat{F}_{-n} \). So \( F(x) = \sum_{n \in \mathbb{Z}} \hat{F}_n e^{inx} = 2i \sum_{n=1}^{\infty} \hat{F}_n \sin nx \). So \( F \in L^2[-\pi, \pi] \) can be approximated by \( \{\sin nx : n \geq 1\} \). Which proves the result [Need to normalize by \( \sqrt{2/\pi} \) so that we have an ONB].

(ii) Extend \( f \) to an even function \( G \in L^2[-\pi, \pi] \), and repeat the above procedure.

**Problem 7.4:** (a) \( T(x) = \sum_{n \in \mathbb{Z}} \hat{T}_n e^{inx} \), and \( S(x) = \sum_{n \in \mathbb{Z}} \hat{S}_n e^{inx} \), where

\[
\hat{T}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| e^{-inx} \, dx
\]

\[
= \frac{1}{2\pi} \int_{0}^{\pi} x [-e^{-inx} + e^{inx}] \, dx
\]

\[
= \frac{1}{\pi} \int_{0}^{\pi} x \cos nx \, dx
\]

\[
= \begin{cases} 
\frac{\pi}{2} & \text{if } n = 0 \\
0 & \text{if } n \text{ is even} \\
-\frac{2}{n^2\pi} & \text{if } n \text{ is odd}
\end{cases}
\]

and

\[
\hat{S}_n = \frac{1}{2\pi} \left[ \int_{0}^{\pi} e^{-inx} \, dx - \int_{-\pi}^{0} e^{-inx} \, dx \right]
\]

\[
= -\frac{i}{\pi} \int_{0}^{\pi} \sin nx \, dx
\]

\[
= \begin{cases} 
0 & \text{if } n = 0 \\
0 & \text{if } n \text{ is even} \\
-\frac{2i}{n\pi} & \text{if } n \text{ is odd}
\end{cases}
\]

(b) Since \( \sum_{n=-\infty}^{\infty} n^2 \hat{T}_n^2 \leq \sum_{n=-\infty, n \neq 0}^{\infty} \frac{1}{n^2} < \infty \), \( T \in H^1(T) \).

Let \( \phi \in C^1(T) \). Then

\[
\int_{-\pi}^{\pi} T(x)\phi'(x) \, dx = -\int_{-\pi}^{0} x\phi'(x) \, dx + \int_{0}^{\pi} x\phi'(x) \, dx
\]
\[
\begin{align*}
&= -\pi \phi(-\pi) + \int_{-\pi}^{0} \phi(x) \, dx + \pi \phi(\pi) - \int_{0}^{\pi} \phi(x) \, dx \\
&= -\int_{-\pi}^{\pi} S(x) \phi(x) \, dx \quad (\text{since } \phi(-\pi) = \phi(\pi)).
\end{align*}
\]
Therefore \( T' = S \).

(c) From part (a), \( \sum_{n \in \mathbb{Z}} n^2 \hat{S}_n^2 = \infty \). So \( S \notin H^1(\mathbb{T}) \).

**Problem 7.5:** Follow the same method as in Lemma 7.8. Let \( M = (M_1, \ldots, M_d) \in \mathbb{N}^d \) and \( N = (N_1, \ldots, N_d) \in \mathbb{N}^d \), and define \( S_M(x) = \sum_{-M \leq k \leq M} a_n e^{ikx} \), where \(-M \leq k \leq M\) indicates that \(-m_i \leq k_i \leq m_i \; \forall 1 \leq i \leq d\). Then
\[
\|S_N - S_M\|_\infty \leq \sum_{M < n \leq N} |a_n| \leq \sum_{M < n \leq N} |n|^k |a_n| \frac{1}{|n^k|} \leq \left[ \sum_{M \leq n \leq N} |n|^{2k} |a_n|^2 \right]^{1/2} \left[ \sum_{M \leq n \leq N} \frac{1}{|n|^{2k}} \right]^{1/2} \leq C \left[ \sum_{M \leq n \leq N} \frac{1}{|n|^{2k}} \right]^{1/2}.
\]
Let \( m = \min_i M_i \), then
\[
\sum_{M \leq n \leq N} \frac{1}{|n|^{2k}} \leq \int_{m}^{\infty} \int_{|Sd-1|} \frac{1}{r^{2k}} r^{d-1} dS d\gamma = C_0 \int_{m}^{\infty} \frac{1}{r^{2k-d+1}} \, dr = C_0 \frac{1}{m^{2k-d}}.
\]
Which proves that \( \|S_N - S_M\|_\infty \to 0 \) as \( m \to \infty \) i.e, \( M \to \infty \).

**Problem 7.6:**
(i) If \( f \in H^1(\mathbb{T}) \) such that \( \int_{\mathbb{T}} f(x) \, dx = 0 \), then \( \hat{f}_0 = 0 \), and consequently \( \|f\|_2^2 = \sum_{n \in \mathbb{Z}} |\hat{f}_n|^2 \leq \sum_{n \in \mathbb{Z}} n^2 |\hat{f}_n|^2 = \|f'\|_2^2 \).

(ii) If \( f \in H^1([0, \pi]) \) such that \( f(0) = f(\pi) = 0 \), then we can extend \( f \) to an odd function \( F \) on \([-\pi, \pi]\), and thus \( \int_{\mathbb{T}} F(x) \, dx = 0 \). Consequently, \( \|F\|_2^2 \leq \|F'\|_2^2 \). Note that \( \|F\|_2^2 = 2\|f\|_2^2 \), and \( \|F'\|_2 = 2\|f'\|_2^2 \). Therefore \( \|f\|_2^2 \leq \|f'\|_2^2 \).

(iii) In general, if \( f \in H^1([a, b]) \) such that \( f(a) = 0 = f(b) \), then define \( g \in H^1([0, \pi]) \) as \( g(x) = f(a + (x(b-a)/\pi)) \). Note that \( g'(x) = f'(a + (x(b-a)/\pi)) \cdot (b-a)/\pi \). And the result follows from (ii).
Problem 7.7: Without loss of generality let us assume that $L = \pi$. According to the initial condition $f(0) = 0 = f(\pi)$. Extend $f$ to an odd function on $[-\pi, \pi]$. Then $f_n = -f_n$, where $f_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx$. Now following the same method as described in section 7.3, we obtain the solution as

$$u(x, t) = \sum_{n=1}^{\infty} f_n e^{-n^2 t} \left[ e^{inx} - e^{-inx} \right]$$

$$= 2i \sum_{n=1}^{\infty} f_n e^{-n^2 t} \sin nx.$$