MAT 201B — Homework 3

February 8, 2016 — Indrajit Jana

The following solutions may not be complete, and should be treated as rough sketches

Problem 7.1: (a)

$$\begin{aligned}
\frac{1}{c_n} &= 2\int_0^{\pi} 2^n \cos^{2n}\frac{x}{2}\,dx \\
&= 4\int_0^{\pi/2} 2^n \cos^{2n} y\,dy \\
&= 2^{n+1}\frac{\Gamma(1/2)\Gamma((2n+1)/2)}{\Gamma(n+1)} \\
&\sim 2\frac{2^n}{\sqrt{n}} \quad \text{as } n\to\infty, \ \ \text{(by Stirling's approximation).}
\end{aligned}$$

Note that for any $0<\delta<\pi$ if $x\in[-\pi,-\delta]\cup[\delta,\pi]$, then we have $(1+\cos x)\le 1+\cos\delta<2$. Therefore $\frac{\sqrt{n}}{2^n}(1+\cos\delta)^n\to 0$ as $n\to\infty$ i.e, $\phi_n(x)=c_n(1+\cos x)^n\to 0$ for $\delta\le|x|\le\pi$. Also $|\phi_n(x)\mathbf{1}_{\{\delta\le|x|\le\pi\}}|\le 1000\frac{1+\cos\delta}{2}\in L^1[\delta,\pi]$. So by DCT we have $\int_{\delta\le|x|\le\pi}\phi_n(x)\,dx\to 0$.

Alternatively

$$\begin{aligned}
c_n\int_{\delta\le|x|\le\pi}(1+\cos x)^n\,dx &= \frac{\int_{\delta\le|x|\le\pi}(1+\cos x)^n\,dx}{\int_{-\pi}^{\pi}(1+\cos x)^n\,dx} \\
&\le \frac{\int_{\delta\le|x|\le\pi}(1+\cos \delta)^n\,dx}{\int_{-\pi}^{\pi}(1+\cos x)^n\,dx} \\
&\le \frac{2(\pi-\delta)(1+\cos\delta)^n}{\int_{|x|\le\delta/2}(1+\cos x)^n\,dx} \\
&\le \frac{2(\pi-\delta)(1+\cos\delta)^n}{\delta(1+\cos(\delta/2))^n} \\
&= \frac{\pi-\delta}{\delta}\left(\frac{1+\cos\delta}{1+\cos(\delta/2)}\right)^n\to 0.
\end{aligned}$$

(b) Continuous functions on $\mathbb{T}$ are periodic continuous, and therefore can be uniformly approximated by the trigonometric polynomials. As a consequence (of uniform convergence) we have L^2 convergence.

I think the actual question would be *prove that $\mathcal{P}$ is dense in the space of continuous functions on* $[0,2\pi]$ *with the L^2-norm.*

Let $f\in C([0,2\pi])$. Since f is continuous on a compact interval, there exists $M>0$ such that $|f(x)|\le M$ for all $x\in[0,2\pi]$. Let us take $\epsilon>0$ and define $\delta=\epsilon^2/M^2$. Now define $\tilde f(x)=f(x)\mathbf{1}_{(\delta,2\pi-\delta)}(x)+$

$\frac{f(\delta)}{\delta}x\mathbf{1}_{[0,\delta]}(x) + \frac{f(2\pi-\delta)}{\delta}(2\pi - x)\mathbf{1}_{[2\pi-\delta,2\pi]}(x)$. *(i.e, $\tilde{f} = f$ on $[\delta, 2\pi - \delta]$ and then linearly join to 0 after that).* Notice that $\|f - \tilde{f}\|_2 < 2\epsilon$.

Also $\tilde{f}(0) = 0 = \tilde{f}(2\pi)$, i.e., it is periodic continuous on $\mathbb{T}$ and hence can be uniformly approximated by elements in $\mathcal{P}$. Let $\phi \in \mathcal{P}$ such that $\|\phi - \tilde{f}\|_\infty < \epsilon$. Then $\|\phi - f\|_2 \le \|\phi - \tilde{f}\|_2 + \|\tilde{f} - f\|_2 < \sqrt{2\pi}\epsilon + 2\epsilon$. Hence $\mathcal{P}$ is dense in $C([0, 2\pi])$.

(c) If $f(0) \neq f(2\pi)$, then it is discontinuous as a function on $\mathbb{T}$. And hence can not be uniformly approximated by continuous functions on $\mathbb{T}$.

Problem 7.2: (a) Note that

$$\begin{aligned}
\sum_{n=-N}^{N} e^{inx} &= e^{-iNx}\frac{1 - e^{i(2N+1)x}}{1 - e^{ix}} \\
&= e^{-iNx}\frac{1 - \cos(2N+1)x - i\sin(2N+1)x}{1 - \cos x - i\sin x} \\
&= e^{-iNx}\frac{2\sin(N+1/2)x[\sin(N+1/2)x - i\cos(N+1/2)x]}{2\sin(x/2)[\sin(x/2) - i\cos(x/2)]} \\
&= 2\pi D_n(x)e^{-iNx}\frac{-ie^{i(N+1/2)x}}{-ie^{ix/2}} \\
&= 2\pi D_n(x).
\end{aligned}$$

Therefore

$$\begin{aligned}
S_N &= \frac{1}{\sqrt{2\pi}}\sum_{n=-N}^{N} \hat{f}_n e^{inx} \\
&= \frac{1}{2\pi}\sum_{n=-N}^{N}\int_{\mathbb{T}} f(y)e^{in(x-y)}\,dy \\
&= (D_N * f)(x).
\end{aligned}$$

(b) Note that

$$\begin{aligned}
\sum_{n=0}^{N}\sin(n+1/2)x &= \frac{1}{2i}\sum_{n=0}^{N}[\exp\{i(n+1/2)x\} - \exp\{-i(n+1/2)x\}] \\
&= \frac{1}{2i}\left[\frac{1 - \exp\{i(N+1)x\}}{\exp\{-ix/2\} - \exp\{ix/2\}} + \frac{1 - \exp\{-i(N+1)x\}}{\exp\{-ix/2\} - \exp\{ix/2\}}\right] \\
&= \frac{1}{2i}\frac{4\sin^2(N+1)x/2}{-2i\sin(x/2)} \\
&= \frac{\sin^2(N+1)x/2}{\sin(x/2)}.
\end{aligned}$$

Therefore $T_N = \frac{1}{N+1}\left(\sum_{n=0}^{N} D_n\right) * f = F_N * f$, where

$$F_N(x) = \frac{1}{2\pi(N+1)}\left(\frac{\sin(N+1)x/2}{\sin(x/2)}\right)^2.$$

(c) D_N could be negative (e.g. $D_1(\pi) = -1/2\pi$), and thus can not be approximate identities.
On the other hand,

(i) $F_N(x) \geq 0$.

(ii) Notice from the description of T_N and S_N

$$\int_{-\pi}^{\pi} F_N(x)\,dx = \frac{1}{2\pi(N+1)} \sum_{n=-N}^{N} (N+1-n) \int_{-\pi}^{\pi} e^{inx}\,dx = 1.$$

(iii) For any $\delta > 0$, $F_N(x) \leq 1/[2\pi(N+1)\sin^2(\delta/2)]$ for all $\delta \leq |x| \leq \pi$. It follows that $\int_{\delta\leq|x|\leq\pi} F_N(x)\,dx \to 0$.

So $\{F_N\}$ are approximation to identity.

From theorem 7.2, $T_N \to f$ uniformly. However S_N may not converge uniformly.

Problem 7.3: We know that $\{e^{inx} : n \in \mathbb{Z}\}$ forms an orthogonal basis of $L^2[-\pi,\pi]$. Let $f \in L^2[0,\pi]$.

(i) Extend f to an odd function $F \in L^2[-\pi,\pi]$. Say $F(x) = f(x)\mathbf{1}_{[0,\pi]}(x) - f(-x)\mathbf{1}_{[-\pi,0]}(x)$. Notice that $\hat{F_n} = -\hat{F_{-n}}$. So $F(x) = \sum_{n\in\mathbb{Z}} \hat{F_n} e^{inx} = 2i\sum_{n=1}^{\infty} \hat{F_n}\sin nx$. So $F \in L^2[-\pi,\pi]$ can be approximated by $\{\sin nx : n \geq 1\}$. In particular, $f = F|_{[0,\pi]}$ can also be approximated by $\{\sin nx : n \geq 1\}$. Which proves the result *[Need to normalize by $\sqrt{2/\pi}$ so that we have an ONB]*.

(ii) Extend f to an even function $G \in L^2[-\pi,\pi]$, and repeat the above procedure.

Problem 7.4: (a) $T(x) = \sum_{n\in Z} \hat{T_n} e^{inx}$, and $S(x) = \sum_{n\in\mathbb{Z}} \hat{S_n} e^{inx}$, where

$$\begin{aligned}
\hat{T_n} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{0}^{\pi} x[e^{-inx} + e^{inx}]\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx \\
&= \begin{cases} \frac{\pi}{2} & \text{if } n = 0 \\ 0 & \text{if } n \text{ is even} \\ -\frac{2}{n^2\pi} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\hat{S_n} &= \frac{1}{2\pi}\left[\int_{0}^{\pi} e^{-inx}\,dx - \int_{-\pi}^{0} e^{-inx}\,dx\right] \\
&= -\frac{i}{\pi}\int_{0}^{\pi} \sin nx\,dx \\
&= \begin{cases} 0 & \text{if } n = 0 \\ 0 & \text{if } n \text{ is even} \\ -\frac{2i}{n\pi} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

(b) Since $\sum_{n=-\infty}^{\infty} n^2\hat{T}_n^2 \leq \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{n^2} < \infty$, $T \in H^1(\mathbb{T})$.

Let $\phi \in C^1(\mathbb{T})$. Then

$$\int_{-\pi}^{\pi} T(x)\phi'(x)\,dx = -\int_{-\pi}^{0} x\phi'(x)\,dx + \int_{0}^{\pi} x\phi'(x)\,dx$$

$$
\begin{aligned}
&= -\pi\phi(-\pi) + \int_{-\pi}^{0} \phi(x)\, dx + \pi\phi(\pi) - \int_{0}^{\pi} \phi(x)\, dx \\
&= -\int_{-\pi}^{\pi} S(x)\phi(x)\, dx \quad (\text{since } \phi(-\pi) = \phi(\pi)).
\end{aligned}
$$

Therefore $T' = S$.

(c) From *part (a)*, $\sum_{n\in\mathbb{Z}} n^2 \hat{S}_n^2 = \infty$. So $S \notin H^1(\mathbb{T})$.

Problem 7.5: *Follow the same method as in Lemma 7.8.* Let $M = (M_1, \ldots, M_d) \in \mathbb{N}^d$ and $N = (N_1, \ldots, N_d) \in \mathbb{N}^d$, and define $S_M(x) = \sum_{-M\le k\le M} a_n e^{ik\cdot x}$, where $-M \le k \le M$ indicates that $-m_i \le k_i \le m_i \; \forall\, 1 \le i \le d$. Then

$$
\begin{aligned}
\|S_N - S_M\|_\infty &\le \sum_{M<n\le N} |a_n| \\
&= \sum_{M<n\le N} |n|^k |a_n| \frac{1}{|n^k|} \\
&\le \left[\sum_{M<n\le N} |n|^{2k}|a_n|^2\right]^{1/2} \left[\sum_{M<n\le N} \frac{1}{|n|^{2k}}\right]^{1/2} \\
&= C\left[\sum_{M<n\le N} \frac{1}{|n|^{2k}}\right]^{1/2}.
\end{aligned}
$$

Let $m = \min_i M_i$, then

$$
\begin{aligned}
\sum_{M<n\le N} \frac{1}{|n|^{2k}} &\le \int_m^\infty \int_{S^{d-1}} \frac{1}{r^{2k}}\, r^{d-1} dS dr \\
&= C_0 \int_m^\infty \frac{1}{r^{2k-d+1}}\, dr \\
&= C_1 \frac{1}{m^{2k-d}}.
\end{aligned}
$$

Which proves that $\|S_N - S_M\|_\infty \to 0$ as $m \to \infty$ i.e, $M \to \infty$.

Problem 7.6: (i) If $f \in H^1(\mathbb{T})$ such that $\int_{\mathbb{T}} f(x)\, dx = 0$, then $\hat{f}_0 = 0$, and consequently $\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}_n|^2 \le \sum_{n\in\mathbb{Z}} n^2|\hat{f}_n|^2 = \|f'\|_2^2$.

(ii) If $f \in H^1([0,\pi])$ such that $f(0) = f(\pi) = 0$, then we can extend f to an odd function F on $[-\pi,\pi]$, and thus $\int_{\mathbb{T}} F(x)\, dx = 0$. Consequently, $\|F\|_2^2 \le \|F'\|_2^2$. Note that $\|F\|_2^2 = 2\|f\|_2^2$, and $\|F'\|_2 = 2\|f'\|_2^2$. Therefore $\|f\|_2^2 \le \|f'\|_2^2$.

(iii) In general, if $f \in H^1([a,b])$ such that $f(a) = 0 = f(b)$, then define $g \in H^1([0,\pi])$ as $g(x) = f(a + (x(b-a)/\pi))$. Note that $g'(x) = f'(a + (x(b-a)/\pi)) \cdot (b-a)/\pi$. And the result follows from *(ii)*.

Problem 7.7: Without loss of generality let us assume that $L = \pi$. According to the initial condition $f(0) = 0 = f(\pi)$. Extend f to an odd function on $[-\pi, \pi]$. Then $f_{-n} = -f_n$, where $f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$. Now following the same method as described in section 7.3, we obtain the solution as

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} f_n e^{-n^2 t}\left[e^{inx} - e^{-inx}\right] \\ &= 2i\sum_{n=1}^{\infty} f_n e^{-n^2 t} \sin nx. \end{aligned}$$