The following solutions may not be complete, and should be treated as rough sketches.

**Problem 7.9:**

\[
(u^2_t + c^2 u_x^2)_t - (2c^2 u_t u_x)_x = 2u_t u_{tt} + 2c^2 u_x u_{tx} - 2c^2 u_{tx} u_x - 2c^2 u_t u_{xx}
\]

\[= 0\]

\[u(0, t) = 0 = (1, t)\) implies that \(u_t(0, t) = 0 = u_t(1, t)\). Consequently form the above equation, \(\int_0^1 (u^2_t + c^2 u_x^2)_t \, dx = 0\) and thus \(\int_0^1 (u^2_t + c^2 u_x^2)_t \, dx = \) constant.

**Problem 7.10:** Simple direct verification.

**Problem 7.14:** Denote \(f(\theta) = \sin^2 \theta\), \(S(\theta) = 2\theta \mod 2\pi\), \(T(x) = 4x(1 - x)\).

**First Part:** If \(\mu = 1\), then a simple calculation will show that \(4\sin^2 \theta_n (1 - \cos^2 \theta_n) = \sin^2 2\theta_n\).

**Second part:** (a) **Orbits:** [The question asks “what can you say about the orbits?”]. It did not ask me to find the orbits. Let us denote the orbit of \(x\) under the transformation \(T\) by \(\mathcal{O}_T(x)\) i.e., \(\mathcal{O}_T(x) = \{x\} \cup \{T^n(x) : n \in \mathbb{N}\}\). Notice that \(\mathcal{O}_T(0) = \{0\}\), \(\mathcal{O}_T(1) = \{1, 0\}\), \(\mathcal{O}_T(1/2) = \{1/2, 1, 0\}\). So we observe that if the trajectory of \(x\) ever hits 1/2, then in the next steps it looks like 1/2 → 1 → 0, and stays at 0 forever.

**Observation 2:** Let us look at the transformation \(S : [0, 2\pi] \rightarrow [0, 2\pi]\) defined as \(S(\theta) \equiv 2\theta \mod 2\pi\). According to the (generalized) Fermat’s little theorem any odd number \((2k + 1)\) divides \(2^p(2^{k+1}) - 1\), where \(\phi\) is the Euler’s phi function. Therefore for any \(m \in \mathbb{N}\), there exists \(q \in \mathbb{N}\) such that \(2(2^q - 1)\) is divisible by \(m\). And thus for any rational \(\frac{p}{m} \in \mathbb{Q}\), \(2^{p+q} \cdot \frac{p}{m} \equiv 2^p \cdot \frac{p}{m} \mod 2\pi\) for some \(p, q \in \mathbb{N}\). Which show that the \(\mathcal{O}_S(\alpha \pi)\) is finite iff \(\alpha \in \mathbb{Q}\).

**Observation 3:** We can transform the above information to the space \([0, 1]\) via the map \(f(\theta) = \sin^2 \theta\). So we conclude that \(\mathcal{O}_T(x)\) is finite iff \(x = \sin^2 2\alpha \pi\) for some rational \(\alpha \in \mathbb{Q}\).

(b) **Invariant measure:** It is easy to see that the Lebesgue measure \(\mathcal{L}\) on \([0, 2\pi]\) is invariant under the transformation \(S\).

Now we’ll construct an invariant measure for \(T\). Consider the measure \(\mathcal{P}\) on \([0, 1]\) defined as

\[
\mathcal{P}([a, b]) = \int_a^b \frac{1}{2\sqrt{x(1-x)}} \, dx = \sin^{-1} \sqrt{b} - \sin^{-1} \sqrt{a}.
\]  

(1)

Let \(t^- \in [0, 1]\) be the smallest root of the quadratic equation \(T(x) = t\).

[Note that if \(T(t^-) = t = T(t^+),\) then \(1/2 - t^- = t^+ - 1/2\). Also a simple calculation would yield that \(\sin^{-1} \sqrt{t} = 2\sin^{-1} \sqrt{T^-}\).]

The graph of \(T\) is symmetric with respect to \(x = 1/2\). So a simple picture would yield

\[
\mathcal{P}(T^{-1}[a, b]) = 2\mathcal{P}([a^-, b^-])
\]
\begin{align*}
&= 2 \int_{a}^{b} \frac{1}{2 \sqrt{x(1-x)}} \, dx \\
&= 2 \sin^{-1} \sqrt{b} - 2 \sin^{-1} \sqrt{a} \\
&= \sin^{-1} \sqrt{b} - \sin^{-1} \sqrt{a} \\
&= \mathcal{P}([a, b]).
\end{align*}

So the measure \( \mathcal{P} \) on \([0, 1]\) is invariant under \( T \).

**Motivation:** The above solution may look not so intuitive. However here is the motivation of the construction of the measure \( \mathcal{P} \).

*Observation 1:* The Lebesgue measure \( \mathcal{L} \) on \([0, 2\pi]\) is invariant under the transformation \( S(\theta) = 2\theta \).

*Observation 2:* \((T \circ f)(\theta) = 4 \sin^2 \theta (1 - \sin^2 \theta) = \sin^2 2\theta = (f \circ S)(\theta) \). So we have the following picture.

[In the picture, \( \mu \) is same as the \( \mathcal{P} \) (the picture was drawn before writing up the solution).] Now we are going to use the invariance of the Lebesgue measure \( \mathcal{L} \) on \([0, 2\pi]\) to construct an invariant measure \( \mathcal{P} \) on \([0, 1]\).

Let \( \mathcal{P} \) be a measure on \([0, 1]\) such that

\[ \mathcal{P}(B) = \mathcal{L}(f^{-1}(B)). \tag{2} \]

Then for any measurable set \( B \subset [0, 1] \),

\[
\mathcal{P}(T^{-1}(B)) = \mathcal{L}(f^{-1}(T^{-1}(B))) \\
= \mathcal{L}((T \circ f)^{-1}(B)) \\
= \mathcal{L}(f \circ S)^{-1}(B) \\
= \mathcal{L}(S^{-1}(f^{-1}(B))) \\
= \mathcal{L}(f^{-1}(B)) \text{ (since } \mathcal{L} \text{ is invariant under } S) \\
= \mathcal{P}(B).
\]
From the discussion about the orbits, we hope to have ergodic theorem for irrational \( \alpha \). Letting \( u \) be the beginning point. Note that \( \psi \) is invariant under \( T \). Since \( \psi \) are the simple functions supported on disjoint intervals, they are orthogonal to each other. Notice \( \alpha \left\lfloor \frac{k}{2^n} \right\rfloor \) happens only for null set of full set. So \( \alpha \) is ergodic. We notice that \( \alpha = \frac{1}{2}(1 + \sqrt{5}) \), then \( 1 - \alpha = \frac{1}{2}(1 - \sqrt{5}) = -\frac{1}{\alpha} \). Using these relations and letting \( u_n = \alpha^n + (1 - \alpha)^n \), we have

\[
\begin{align*}
u_n + u_{n-1} &= \alpha^n + (1 - \alpha)^n + \alpha^{n-1} + (1 - \alpha)^{n-1} \\
&= \alpha^{n-1}(1 + \alpha) + (1 - \alpha)^{n-1}(1 - \alpha + 1) \\
&= \alpha^{n-1} \left( 1 - \frac{1}{1 - \alpha} \right) + (1 - \alpha)^{n-1} \left( -\frac{1}{\alpha} + 1 \right) \\
&= -\frac{\alpha^n}{1 - \alpha} \left( 1 - \frac{-(1 - \alpha)^n}{\alpha} \right) \\
&= \alpha^{n+1} + (1 - \alpha)^{n+1} \\
&= u_{n+1}.
\end{align*}
\]

Note that \( u_0 = 2, u_1 = 1 \), and hence all \( u_n \) are integers. We also notice that \( (1 - \alpha)^n \to 0 \) as \( n \to \infty \). Therefore \( \alpha^n \) is getting close to \( u_n \), which is an integer. As a result, \( \lim_{n\to\infty}(\alpha^n \mod 1) = 0 \). In other words, \( \lim_{n\to\infty}(x_n \mod 1) = 0 \).

Problem 7.15: We notice that

\[
\psi_{n,k} = \begin{cases} 
2n/2 & \text{if } k/2^n \leq x < k/2^n + 1/2^{n+1} \\
-2n/2 & \text{if } k/2^n + 1/2^{n+1} \leq x < (k + 1)/2^n \\
0 & \text{otherwise.}
\end{cases}
\]

Since \( \psi_{n,k} \) are the simple functions supported on disjoint intervals, they are orthogonal to each other. Notice that \( \int_0^1 \psi_{n,k}(x) \, dx = 0 \), and for \( m < n \), \( \psi_{m,*}(x) \) are constant on the support of \( \phi_{n,*}(x) \). So \( \langle \psi_{n,i}, \psi_{n,k} \rangle = 0 \) for \( m \neq n \). Therefore \( \bigcup_{n=1}^N B_n \) is an orthogonal set of vectors. We have

\[
V_N = \{ f : f \text{ is constant on } [k/2^n, (k + 1)/2^n) \text{ for all } 0 \leq k \leq 2^N - 1 \}.
\]
Dimension of $V_N$ is the same as that of $\bigcup_{n=0}^N B_n$.

**Problem 7.18:**

(a) Simple change of variables, \( \int_a^{a+1} e_n(x-a)e_m(x-a) \, dx = \int_0^1 e_n(x)e_m(x) \, dx = \delta_{m,n} \). Similarly, show the completeness.

(b) Similar to part (a).

(c) The elements of $B$ are orthonormal among themselves and so are the elements of $B_1$. Due to having disjoint support, the elements from $B$ are orthonormal to the elements of $B_1$.

\[ f(x) = f_1(x) + f_2(x) = \sum_{i=1}^{\infty} \langle e_n, f_1 \rangle e_n(x) + \sum_{i=1}^{\infty} \langle e_n, f_2 \rangle e_n(x) \] (all the equalities are in $L^2$ sense), where $e_n(x) = e_n(x-1)$.

(d) Similar to part (c)