MAT 201B — Homework 4

February 16, 2016 — Indrajit Jana

THE FOLLOWING SOLUTIONS MAY NOT BE COMPLETE, AND SHOULD BE TREATED AS ROUGH SKETCHES

Problem 7.9:

$$\begin{aligned}(u_t^2 + c^2u_x^2)_t - (2c^2u_tu_x)_x &= 2u_tu_{tt} + 2c^2u_xu_{tx} - 2c^2u_{xt}u_x - 2c^2u_tu_{xx} \\ &= 0\end{aligned}$$

$u(0,t) = 0 = (1,t)$ implies that $u_t(0,t) = 0 = u_t(1,t)$. Consequently form the above equation, $\int_0^1 (u_t^2 + c^2u_x^2)_t \, dx = 0$ and thus $\int_0^1 (u_t^2 + c^2u_x^2)_t \, dx$ =constant.

Problem 7.10: Simple direct verification.

Problem 7.14: Denote $f(\theta) = \sin^2\theta,\ S(\theta) = 2\theta \bmod 2\pi,\ T(x) = 4x(1-x)$.

First Part: If $\mu = 1$, then a simple calculation will show that $4\sin^2\theta_n(1-\cos^2\theta_n) = \sin^2 2\theta_n$.

Second part: (a)*Orbits:* *[The question asks "what can you say about the orbits?". It did not ask me to find the orbits]* Let us denote the orbit of x under the transformation T by $\mathcal{O}_T(x)$ i.e., $\mathcal{O}_T(x) = \{x\} \cup \{T^n(x) : n \in \mathbb{N}\}$. Notice that $\mathcal{O}_T(0) = \{0\}, \mathcal{O}_T(1) = \{1,0\}, \mathcal{O}_T(1/2) = \{1/2,1,0\}$. So we observe that if the trajectory of x ever hits $1/2$, then in the next steps it looks like $1/2 \to 1 \to 0$, and stays at 0 forever.

Observation 2: Let us look at the transformation $S : [0,2\pi] \to [0,2\pi]$ defined as $S(\theta) \equiv 2\theta \bmod 2\pi$. According to the (generalized) Fermat's little theorem, any odd number $(2k+1)$ divides $2^{\phi(2k+1)} - 1$, where ϕ is the Euler's phi function. Therefore for any $m \in \mathbb{N}$, there exists $q \in \mathbb{N}$ such that $2(2^q - 1)$ is divisible by m. And thus for any rational $\frac{n}{m} \in \mathbb{N}$, $2^{p+q}\frac{n}{m}\pi \equiv 2^p\frac{n}{m}\pi \bmod 2\pi$ for some $p,q \in \mathbb{N}$. Which show that the $\mathcal{O}_S(\alpha\pi)$ is finite iff $\alpha \in \mathbb{Q}$.

Observation 3: We can transform the above information to the space $[0,1]$ via the map $f(\theta) = \sin^2\theta$. So we conclude that $\mathcal{O}_T(x)$ is finite iff $x = \sin^2 2\alpha\pi$ for some rational $\alpha \in \mathbb{Q}$.

(b)*Invariant measure:* It is easy to see that the Lebesgue measure $\mathcal{L}$ on $[0,2\pi]$ is invariant under the transformation S.

Now we'll construct an invariant measure for T. Consider the measure $\mathcal{P}$ on $[0,1]$ defined as

$$\mathcal{P}([a,b]) = \int_a^b \frac{1}{2\sqrt{x(1-x)}}\, dx = \sin^{-1}\sqrt{b} - \sin^{-1}\sqrt{a}. \tag{1}$$

Let $t^- \in [0,1]$ be the smallest root of the quadratic equation $T(x) = t$.

[Note that if $T(t^-) = t = T(t^+)$, then $1/2 - t^- = t^+ - 1/2$. Also a simple calculation would yield that $\sin^{-1}\sqrt{t} = 2\sin^{-1}\sqrt{t^-}$].

The graph of T is symmetric with respect to $x = 1/2$. So a simple picture would yield

$$\mathcal{P}(T^{-1}[a,b]) = 2\mathcal{P}([a^-,b^-])$$

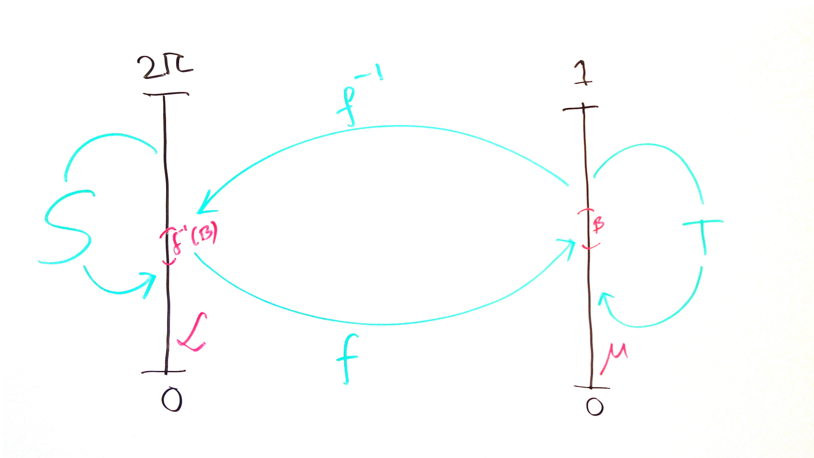
$$
\begin{aligned}
&= 2\int_{a^-}^{b^-} \frac{1}{2\sqrt{x(1-x)}}\, dx \\
&= 2\sin^{-1}\sqrt{b^-} - 2\sin^{-1}\sqrt{a^-} \\
&= \sin^{-1}\sqrt{b} - \sin^{-1}\sqrt{a} \\
&= \mathcal{P}([a,b]).
\end{aligned}
$$

So the measure $\mathcal{P}$ on $[0,1]$ is invariant under T.

Motivation: The above solution may look not so intuitive. However here is the motivation of the construction of the measure $\mathcal{P}$.

<u>*Observation 1:*</u> The Lebesgue measure $\mathcal{L}$ on $[0,2\pi]$ is invariant under the transformation $S(\theta) = 2\theta$.

<u>*Observation 2:*</u> $(T \circ f)(\theta) = 4\sin^2\theta(1-\sin^2\theta) = \sin^2 2\theta = (f \circ S)(\theta)$. So we have the following picture.



[In the picture, μ is same as the $\mathcal{P}$ (the picture was drawn before writing up the solution).] Now we are going to use the invariance of the Lebesgue measure $\mathcal{L}$ on $[0,2\pi]$ to construct an invariant measure $\mathcal{P}$ on $[0,1]$.

Let $\mathcal{P}$ be a measure on $[0,1]$ such that

$$\mathcal{P}(B) = \mathcal{L}(f^{-1}(B)). \tag{2}$$

Then for any measurable set $B \subset [0,1]$.

$$
\begin{aligned}
\mathcal{P}(T^{-1}(B)) &= \mathcal{L}(f^{-1}(T^{-1}(B))) \\
&= \mathcal{L}((T\circ f)^{-1}(B)) \\
&= \mathcal{L}((f\circ S)^{-1}(B)) \\
&= \mathcal{L}(S^{-1}(f^{-1}(B))) \\
&= \mathcal{L}(f^{-1}(B)) \quad \text{(since } \mathcal{L} \text{ is invariant under } S) \\
&= \mathcal{P}(B).
\end{aligned}
$$

Which proves that $\mathcal{P}$ is invariant under T.

Explicit construction of $\mathcal{P}$: The function $f : [0, 2\pi] \to [0, 1]$ defined as $f(\theta) = \sin^2\theta$ is not onto, and for any $a \in [0, 1)$, $f^{-1}(a)$ has four solutions on $[0, 2\pi]$, one on each of $[0, \pi/2), [\pi/2, \pi), [\pi, 3\pi/2), [3\pi/2, 2\pi]$ *[Drawing a picture might be helpful]*. Let $\theta_a \in [0, \pi/2]$ be the smallest solution. Then from our motivation (2) and the symmetry, we have

$$\begin{aligned}
\mathcal{P}([a, b]) &= \mathcal{L}(f^{-1}[a, b]) \\
&= 4\mathcal{L}([\theta_a, \theta_b]) \\
&= 4(\theta_b - \theta_a) \\
&= 4(\sin^{-1}\sqrt{b} - \sin^{-1}\sqrt{a}) \\
&= 4\int_a^b \frac{1}{2\sqrt{x(1-x)}}\,dx,
\end{aligned}$$

which is same as the measure $\mathcal{P}$ that we had used in our solution above (1) *[The constant 4 does not play any role here]*.

(c)*Ergodic theorem:* From the discussion about the orbits, we hope to have ergodic theorem for irrational beginning point. Note that S is ergodic because $S^{-1}([a, b]) = [a, b]$ iff $a = 0, b = 2\pi$. In other words $S^{-1}(E) = E$ happens only for null set of full set. So S is ergodic.

Problem 7.15: Note that if $\alpha = \frac{1}{2}(1 + \sqrt{5})$, then $1 - \alpha = \frac{1}{2}(1 - \sqrt{5}) = -\frac{1}{\alpha}$. Using these relations and letting $u_n = \alpha^n + (1 - \alpha)^n$, we have

$$\begin{aligned}
u_n + u_{n-1} &= \alpha^n + (1-\alpha)^n + \alpha^{n-1} + (1-\alpha)^{n-1} \\
&= \alpha^{n-1}(1+\alpha) + (1-\alpha)^{n-1}(1-\alpha+1) \\
&= \alpha^{n-1}\left(1 - \frac{1}{1-\alpha}\right) + (1-\alpha)^{n-1}\left(-\frac{1}{\alpha} + 1\right) \\
&= -\frac{\alpha^n}{1-\alpha} - \frac{(1-\alpha)^n}{\alpha} \\
&= \alpha^{n+1} + (1-\alpha)^{n+1} \\
&= u_{n+1}.
\end{aligned}$$

Note that $u_0 = 2, u_1 = 1$, and hence all u_n are integers. We also notice that $(1 - \alpha)^n \to 0$ as $n \to \infty$. Therefore α^n is getting close to u_n, which is an integer. As a result, $\lim_{n\to\infty}(\alpha^n \bmod 1) = 0$. In other words, $\lim_{n\to\infty}(x_n \bmod 1) = 0$

Problem 7.17: We notice that

$$\psi_{n,k} = \begin{cases} 2^{n/2} & \text{if } k/2^n \le x < k/2^n + 1/2^{n+1} \\ -2^{n/2} & \text{if } k/2^n + 1/2^{n+1} \le x < (k+1)/2^n \\ 0 & \text{otherwise.} \end{cases}$$

Since $\psi_{n,k}$ are the simple functions supported on disjoint intervals, they are orthogonal to each other. Notice that $\int_0^1 \psi_{n,k}(x)\,dx = 0$, and for $m < n$, $\psi_{m,\cdot}(x)$ are constant on the support of $\phi_{n,\cdot}(x)$. So $\langle \psi_{n,l}, \psi_{n,k} \rangle = 0$ for $m \neq n$. Therefore $\cup_{n=0}^N B_n$ is an orthogonal set of vectors. We have

$$V_N = \{f : f \text{ is constant on } [k/2^N, (k+1)/2^N) \text{ for all } 0 \le k \le 2^N - 1\}.$$

Dimension of V_N is the same as that of $\cup_{n=0}^{N} B_n$.

Problem 7.18:

(a) Simple change of variables, $\int_a^{a+1} e_n(x-a)e_m(x-a)\, dx = \int_0^1 e_n(x)e_m(x)\, dx = \delta_{m,n}$. Similarly, show the completeness.

(b) Similar to *part (a)*.

(c) The elements of B are orthonormal among themselves and so are the elements of B_1. Due to having disjoint support, the elements from B are orthonormal to the elements of B_1.

Any function $f \in L^2[0,2]$ can be written as $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = f(x)\mathbf{1}_{[0,1]}(x)$, and $f_2(x) = f(x)\mathbf{1}_{(1,2]}(x)$. Due to having disjoint support, $\|f\|^2_{L^2[0,2]} = \|f_1\|^2_{L^2[0,1]} + \|f_2\|^2_{L^2[1,2]}$. Now $f(x) = f_1(x) + f_2(x) = \sum_{i=1}^{\infty} \langle e_n, f_1\rangle f_1(x) + \sum_{i=1}^{\infty} \langle e_n^1, f_2\rangle f_2(x)$ *(all the equalities are in L^2 sense)*, where $e_n^1(x) = e_n(x-1)$.

(d) Similar to *part (c)*