MAT 201B **Homework 5**

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THE FOLLOWING SOLUTIONS MAY NOT BE COMPLETE, AND SHOULD BE TREATED AS ROUGH SKETCHES

Problem 8.2: Follows from the definition.

Problem 8.3: *(a) implies (b):* Since $\mathcal{M}, \mathcal{N} \subset \mathcal{H}$ are closed subspaces of $\mathcal{H}$, we can write $\mathcal{M} \oplus \mathcal{M}^\perp = \mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$. So any $x \in \mathcal{H}$ can be decomposed as $x = x_\mathcal{M} + (x - x_\mathcal{M}) = x_\mathcal{N} + (x - x_\mathcal{N})$, where $x_\mathcal{M} \in \mathcal{M}, x_\mathcal{N} \in \mathcal{N}$ and $x_\mathcal{M} \perp (x - x_\mathcal{M}), x_\mathcal{N} \perp (x - x_\mathcal{N})$. Now if $\mathcal{M} \subset \mathcal{N}$, then $Qy = y$ for all $y \in \mathcal{M}$. Which in particular implies that $(QP)x = Q(x_\mathcal{M}) = x_\mathcal{M} = Px$.

(b) implies (a): Let $x \in \mathcal{M}$, then $Px = x$, and thus $Qx = QPx = Px = x$. Therefore $x \in \mathcal{N}$ i.e., $\mathcal{M} \subset \mathcal{N}$.

(a) implies (c): Since $\mathcal{M} \subset \mathcal{N}$, we can write $\mathcal{N} = \mathcal{M} \oplus (\mathcal{M}^\perp \cap \mathcal{N})$. And we also have $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$. Therefore $\mathcal{H} = \mathcal{M} \oplus (\mathcal{M}^\perp \cap \mathcal{N}) \oplus \mathcal{N}^\perp$ and these three components are orthogonal to each other. So any $x \in \mathcal{H}$ can be orthogonally decomposed as $x = x_\mathcal{M} + x_{\mathcal{M}^\perp \cap \mathcal{N}} + x_{\mathcal{N}^\perp}$. Consequently $PQx = P(x_\mathcal{M} + x_{\mathcal{M}^\perp \cap \mathcal{N}}) = x_\mathcal{M} = Px$.

(c) implies (d): Since P, Q are orthogonal projection operators, $\|P\|, \|Q\| \le 1$. Thus for any $x \in \mathcal{H}$, $\|Px\| = \|PQx\| \le \|Qx\|$.

(d) implies (e):

$$\begin{aligned}
\langle x, Px \rangle &= \langle x_\mathcal{M} + (x - x_\mathcal{M}), x_\mathcal{M} \rangle \\
&= \|x_\mathcal{M}\|^2 \\
&\le \|x_\mathcal{N}\|^2 \quad (\text{since } \|x_\mathcal{M}\|^2 = \|Px\|^2 \le \|Qx\|^2 = \|x_\mathcal{N}\|^2) \\
&= \langle x, x_\mathcal{N} \rangle \\
&= \langle x, Qx \rangle.
\end{aligned}$$

(e) implies (a): Let $x \in \mathcal{N}^\perp$, then $Qx = 0$. Consequently, $\|x_\mathcal{M}\|^2 = \langle x, Px \rangle \le \langle x, Qx \rangle = 0$, which implies that $x_\mathcal{M} = 0$ and thus $x \in \mathcal{M}^\perp$. Therefore $\mathcal{N}^\perp \subset \mathcal{M}^\perp$. Since $\mathcal{M}, \mathcal{N}$ are closed, we can conclude that $\mathcal{M} = \mathcal{M}^{\perp\perp} \subset \mathcal{N}^{\perp\perp} = \mathcal{N}$.

Problem 8.4: *First part:* Let $x \in \mathcal{H} = \cup_{n=1}^\infty \mathrm{ran} P_n$. Then there exists $N_x \in \mathbb{N}$ such that $x \in \mathrm{ran} P_n$ for all $n \ge N_x$. Which implies that $x = P_n x$ for all $n \ge N_x$. In other words, $\|P_n x - Ix\| \to 0$ as $n \to \infty$.[1]

Second part: Obviously if $P_n = I$ for all sufficiently large n, then we have $\lim_{n\to\infty} \|P_n - I\|_{op} = 0$. Conversely, if $P_n \ne I$ for all $n \in \mathbb{N}$, then for each n there exists $x_n \in \ker P_n$ such that $\|x_n\| = 1$. In that case,

[1] If $\mathcal{H} = \overline{\cup_{n=1}^\infty \mathrm{ran} P_n}$, then for any $x \in \mathcal{H}$ and any $\epsilon > 0$ we can find $y \in \cup_{n=1}^\infty \mathrm{ran} P_n$ such that $\|x - y\| < \epsilon$. Now there exists $N_y \in \mathbb{N}$ such that $y \in \mathrm{ran} P_n$ for all $n \ge N_y$. Therefore $P_n y = y$ for all $n \ge N_y$, and thus $\|P_n x - x\| = \|P_n x - P_n y\| + \|y - x\| \le (1 + \|P_n\|_{op})\|x - y\| < 2\epsilon$. Which proves that even if $\mathcal{H} = \overline{\cup_{n=1}^\infty \mathrm{ran} P_n}$, $P_n \to I$ strongly.

$\|P_n - I\|_{op} \geq \|P_n x_n - x_n\| = \|x_n\| = 1 \not\to 0$ as $n \to \infty$.

Problem 8.6: <u>*First part:*</u> Obviously if $U : \mathcal{H}_1 \to \mathcal{H}_2$ is unitary then U is an isometric isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$ as a normed linear spaces.

Conversely, suppose $U : \mathcal{H}_1 \to \mathcal{H}_2$ is an isometric isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$ as normed linear spaces. Then isomorphism implies that U is invertible. In addition, the isometry implies that for any $x, y \in \mathcal{H}_1$ we have $\|Ux\|_{\mathcal{H}_2} = \|x\|_{\mathcal{H}_1}, \|Uy\|_{\mathcal{H}_2} = \|y\|_{\mathcal{H}_1}$, and $\|U(x+y)\|_{\mathcal{H}_2} = \|x+y\|_{\mathcal{H}_1}$. Combining all these we obtain

$$\begin{aligned} & \langle U(x+y), U(x+y)\rangle_{\mathcal{H}_2} = \langle x+y, x+y\rangle_{\mathcal{H}_1} \\ \Rightarrow \quad & \langle Ux, Uy\rangle_{\mathcal{H}_2} + \langle Uy, Ux\rangle_{\mathcal{H}_2} = \langle x, y\rangle_{\mathcal{H}_1} + \langle y, x\rangle_{\mathcal{H}_1} \\ \Rightarrow \quad & Re\langle Ux, Uy\rangle_{\mathcal{H}_2} = Re\langle x, y\rangle_{\mathcal{H}_1}. \end{aligned}$$

Similarly, using the fact that $\langle U(x+iy), U(x+iy)\rangle_{\mathcal{H}_2} = \langle x+iy, x+iy\rangle_{\mathcal{H}_1}$, we can prove that $Im\langle Ux, Uy\rangle_{\mathcal{H}_2} = Im\langle x, y\rangle_{\mathcal{H}_1}$. As a result we have

$$\langle Ux, Uy\rangle_{\mathcal{H}_2} = \langle x, y\rangle_{\mathcal{H}_1} \quad \forall\, x, y \in \mathcal{H}_1.$$

<u>*Second part:*</u> $U : \mathcal{H}_1 \to \mathcal{H}_2$ is invertible iff $U^{-1} : \mathcal{H}_2 \to \mathcal{H}_1$ is invertible. Secondly U is unitary iff

$$\begin{aligned} & \langle Ux, Uy\rangle_{\mathcal{H}_2} = \langle x, y\rangle_{\mathcal{H}_1} \quad \forall\, x, y \in \mathcal{H}_1 \\ \Longleftrightarrow \quad & \langle Ux, Uy\rangle_{\mathcal{H}_2} = \langle U^{-1}(Ux), U^{-1}(Uy)\rangle_{\mathcal{H}_1} \quad \forall\, x, y \in \mathcal{H}_1 \\ \Longleftrightarrow \quad & \langle s, t\rangle_{\mathcal{H}_2} = \langle U^{-1}(s), U^{-1}(t)\rangle_{\mathcal{H}_1} \quad \forall\, s, t \in \mathcal{H}_2, \end{aligned}$$

which proves the result.

Problem 8.7: Note that by Cachy-Schwarz inequality, $|\phi_y(x)| = |\langle y, x\rangle| \leq \|x\|\|y\|$. Which implies that $\|\phi_y\| \leq \|y\|$. Secondly, $\phi_y(y/\|y\|) = \|y\|$. Therefore $\|\phi_y\| = \|y\|$.

Problem 8.8: Note that for any $y, z \in \mathcal{H},\ \mu, \lambda \in \mathbb{C}$ we have $\mu\phi_y(x) + \lambda\phi_z(x) = \mu\langle y, x\rangle + \lambda\langle z, x\rangle = \langle \bar{\mu}y + \bar{\lambda}z, x\rangle = \phi_{\bar{\mu}y+\bar{\lambda}z}(x)$. Using this fact, we can validate that the inner product is linear and satisfies all the other properties of inner product. Completeness of $\mathcal{H}^*$ follows directly from the completeness of $\mathcal{H}$.

Problem 8.9: <u>*First part:*</u> Let $\{x_n\}_n, \{y_n\}_n \subset \mathcal{M}$ be two sequences in $\mathcal{M}$ such that $\lim_{n\to\infty} \|x_n - x\| = 0 = \lim_{n\to\infty} \|y_n - x\|$. But since ϕ is bounded, $|\phi(x_n) - \phi(y_n)| \leq \|\phi\|\|x_n - y_n\| \leq \|\phi\|(\|x_n - x\| + \|x - y_n\|) \to 0$ as $n \to \infty$. Therefore $\phi(x) := \lim_{n\to\infty} \phi(x_n)$ is well defined.

<u>*Second part:*</u> If the index set I is countable, then the necessary and sufficient condition is $\sum_\alpha |c_\alpha|^2 < \infty$. Since $\{u_\alpha\}$ is an ONB, any $x \in \mathcal{H}$ can be written as $x = \sum_\alpha \langle u_\alpha, x\rangle u_\alpha$. Now suppose $\sum_\alpha |c_\alpha|^2 < \infty$, then we can construct $c \in \mathcal{H}$ where the αth component of c is c_α w.r.to the basis $\{u_\alpha\}$. And we see that $\phi_c(u_\alpha) = c_\alpha = \phi(u_\alpha)$. Therefore from *problem 8.7*, $\|\phi\| = \|\phi_c\| = \|c\| < \infty$.

Conversely, suppose $\|\phi\| < \infty$. Let us define $C_n = (\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n, 0, 0, 0, \ldots)$ with respect to the basis $\{u_\alpha\}$. Then $\|C_n\|^2 = \sum_{\alpha=1}^n |c_\alpha|^2 = \phi(C_n) \leq \|\phi\|\|C_n\|$. Which implies that $\|C_n\| \leq \|\phi\|$. Since this is true for all n and the right hand side does not depend on n, we can conclude that $\sum_\alpha |c_\alpha|^2 < \infty$.

If the index set I is not countable, then using the *first part* and the above discussion, we can conclude the necessary and sufficient condition is $\sup_{F:\text{finite}} \sum_{F\subset I} |c_\alpha|^2 < \infty$.

Problem 8.11: <u>*First part:*</u> Let $\dim \mathcal{H} = n \geq k = \dim \ker A$. Let $B_k := \{u_1, \dots, u_k\}$ be an ONB of $\ker A$. Extend B_k to an ONB of $\mathcal{H}$ as $B_n = B_k \cup \{u_{k+1}, \dots, u_n\}$. Now obviously, $\text{ran}A = \text{span}\{A(u_i) : 1 \leq i \leq n\}$, but $A(u_i) = 0$ for all $1 \leq i \leq k$. Therefore $\text{ran}A = \text{span}\{A(u_i) : k+1 \leq i \leq n\}$ and thus $\dim\ \text{ran}\ A \leq n-k$. It suffices to show that $A(u_{k+1}), \dots, A(u_n)$ are independent.

Suppose there exists $c_{k+1}, \dots, c_n \in \mathbb{C}$ such that at least one of the $c_i \neq 0$ and $0 = \sum_{i=k+1}^{n} c_i A(u_i) = A\left(\sum_{i=k+1}^{n} c_i u_i\right)$. Then $\sum_{i=k+1}^{n} c_i u_i \in \ker A = \text{span}\{u_1, \dots, u_k\}$. Which contradicts the fact that $B_n = \{u_1, \dots, u_n\}$ is an ONB of $\mathcal{H}$. Therefore $A(u_{k+1}), \dots, A(u_n)$ are independent and thus $\dim\ \text{ran}\ A = n-k = \dim \mathcal{H} - \dim \ker A$.

<u>*Second part:*</u> Notice that for any $x \in \ker A$, and any $y \in \mathcal{H}$ we have $0 = \langle y, Ax\rangle = \langle A^* y, x\rangle$. Which implies that $\ker A = (\text{ran}\ A^*)^\perp$. Therefore $\dim \ker A = \dim(\text{ran}\ A^*)^\perp = \dim \mathcal{H} - \dim \text{ran}A^* = \dim \ker A^*$.

Problem 8.12: Since $A = A^*$, from the *second part of problem 8.11*, we obtain $\ker A = (\text{ran}A)^\perp$. Now if $x \in \ker A$, then we have $\|x\| \leq \frac{1}{c}\|Ax\| = 0$ i.e., $x = 0$. In other words $\ker A = \{0\} = (\text{ran}A)^\perp$. Therefore $\mathcal{H} = \{0\}^\perp = (\text{ran}A)^{\perp\perp} = \overline{\text{ran}A}$. It suffices to show that $\text{ran}A = \overline{\text{ran}A}$. Let $\{Ax_n\}_n$ be a Cauchy sequence in $\text{ran}A$, then $\|x_n - x_m\| \leq \frac{1}{c}\|Ax_n - Ax_m\| \to 0$ as $n \to \infty$. Which implies that $\{x_n\}_n$ is Cauchy in $\mathcal{H}$, and thus $x_n \to x \in \mathcal{H}$ as $n \to \infty$. Since A is bounded $\|Ax_n - Ax\| \to 0$. Which proves that $\text{ran}A$ is closed.

Let $x_1, x_2 \in \mathcal{H}$ such that $Ax_1 = y = Ax_2$. Then $\|x_1 - x_2\| \leq \frac{1}{c}\|Ax_1 - Ax_2\| = 0$. Which implies that $x_1 = x_2$ i.e,. the solution of $Ax = y$ is unique.