MAT 201B Homework 6

March 7, 2016 Indrajit Jana

THE FOLLOWING SOLUTIONS MAY NOT BE COMPLETE, AND SHOULD BE TREATED AS ROUGH SKETCHES

Problem 8.1: (a) Clearly, if $x+M, y+M \in X\backslash M$, then for any $\lambda, \mu \in \mathbb{C}$ we have

$$\lambda(x+M)+\mu(y+M)=(\lambda x+\mu y)+M \in X\backslash M.$$

Therefore $X\backslash M$ is a linear space.

(b) Define the function $\phi : N \to X\backslash M$ as $\phi(x)=x+M$. It is easy to see that ϕ is a linear isomorphism.

(c) This is not true in general. For example, we can take an unbounded linear functional $\phi : X \to \mathbb{C}$. Then the co-dimension of $M=\ker\phi$ is one in X. But ϕ being unbounded, $\ker\phi$ is not closed in X. Construction of such functional can be done using the axiom of choice and Hamel basis.

Problem 8.10: From problem 8.8, we conclude that $\{\phi_{u_\alpha}\}$ is an orthogonal set of vectors. Secondly, for any $u \in \mathcal{H}$ we notice that

$$\begin{aligned}\langle \phi_u, x\rangle &= \langle u, x\rangle \\ &= \left\langle \sum_\alpha \langle u, u_\alpha\rangle u_\alpha, x \right\rangle \\ &= \sum_\alpha \overline{\langle u, u_\alpha\rangle}\langle u_\alpha, x\rangle \\ &= \sum_\alpha \langle \phi_u, \phi_{u_\alpha}\rangle \phi_{u_\alpha}(x).\end{aligned}$$

Therefore $\{\phi_{u_\alpha}\}$ is an ONB of $\mathcal{H}^*$.

Problem 8.13: If $\{u_\alpha\}$ is an orthogonal set. Then $\{u_\alpha\}$ is an ONB iff for any $x \in \mathcal{H}$,

$$\begin{aligned} & x = \sum_\alpha \langle x, u_\alpha\rangle u_\alpha \\ \iff \qquad & I(x) = \sum_\alpha (u_\alpha \otimes u_\alpha)(x).\end{aligned}$$

In other words, $\sum_\alpha u_\alpha \otimes u_\alpha = I$.

Problem 8.14: The given condition implies that $\langle x,(A-B)y\rangle = 0$ for all $x, y \in \mathcal{H}$. Therefore $(A-B)y=0$ for all $y \in \mathcal{H}$ i.e., $A=B$.

For any $x, y \in \mathcal{H}$ we have

$$\begin{aligned} & \langle x+y,(A-B)(x+y)\rangle = 0 \\ \textit{i.e.,} \qquad & \langle x,(A-B)x\rangle + \langle y,(A-B)y\rangle + \langle x,(A-B)y\rangle + \langle y,(A-B)x\rangle = 0 \\ \textit{i.e.,} \qquad & \langle x,(A-B)y\rangle + \langle y,(A-B)x\rangle = 0.\end{aligned}$$

Similarly, $\langle x+iy,(A-B)(x+iy)\rangle = 0$ gives us $\langle x,(A-B)y\rangle - \langle y,(A-B)x\rangle = 0$. As a result we have $\langle x,(A-B)y\rangle = 0$. So from the previous part, $A = B$.

If $\mathcal{H}$ is a real Hilbert space, then we can only have $\langle x,(A-B)y\rangle + \langle y,(A-B)x\rangle = 0$. Which implies that $A-B = -(A-B)^*$. Now if both A and B are self adjoint then we can have $\langle x,(A-B)y\rangle = 0$ i.e., $A = B$.

Problem 8.15: (a) $\langle x, A^{**}y\rangle = \langle A^*x, y\rangle = \langle x, Ay\rangle$.
(b) $\langle x,(AB)^*y\rangle = \langle ABx, y\rangle = \langle Bx, A^*y\rangle = \langle x, B^*A^*y\rangle$.
(c) $\langle x,(\lambda A)^*y\rangle = \langle \lambda Ax, y\rangle = \bar{\lambda}\langle Ax, y\rangle = \langle x, \bar{\lambda}A^*y\rangle$.
(d) $\langle x,(A+B)^*y\rangle = \langle (A+B)x, y\rangle = \langle Ax, y\rangle + \langle Bx, y\rangle = \langle x, A^*y\rangle + \langle x, B^*y\rangle = \langle x,(A^*+B^*)y\rangle$.
(e) Note that if $x, y \in \mathcal{H}$, such that $x \neq 0$ and $\|y\| = 1$ then $|\langle x,y\rangle| \leq \|x\|$, and $|\langle x, x/\|x\|\rangle| = \|x\|$. Therefore we can conclude that

$$\|x\| = \sup_{y:\|y\|=1} |\langle x,y\rangle| = \sup_{y:\|y\|=1} |\langle y,x\rangle|.$$

Using the above, we have

$$\begin{aligned}\|A\| &= \sup_{x:\|x\|=1} \|Ax\| \\ &= \sup_{x:\|x\|=1} \sup_{y:\|y\|=1} |\langle Ax,y\rangle| \\ &= \sup_{x:\|x\|=1} \sup_{y:\|y\|=1} |\langle x,A^*y\rangle| \\ &= \sup_{y:\|y\|=1} \|A^*y\| \\ &= \|A^*\|.\end{aligned}$$

Problem 8.16: It was already shown in the book that $\langle Uf, Ug\rangle = \langle f, g\rangle$ for all $f, g \in L^2(\Omega, P)$. Also for any $f \in L^2(\Omega, P)$, $U(f \circ T^{-1}) = f$. Therefore U is invertible.

Problem 8.17: Let $\{x_n\}_n \subset \mathcal{H}$ be a sequence such that $\|x_n - x\| \to 0$ for some $x \in \mathcal{H}$. Then for any $y \in \mathcal{H}$ we have $|\langle x_n, y\rangle - \langle x, y\rangle| \leq \|x_n - x\|\|y\| \to 0$ as $n \to \infty$. Which implies that $x_n \rightharpoonup x$.

Let $\dim \mathcal{H} = k < \infty$, and $\{u_1, \ldots, u_k\}$ be an ONB of $\mathcal{H}$. Now if $x_n \rightharpoonup x$, then we have $\langle x_n, u_i\rangle \to \langle x, u_i\rangle$ for all $1 \leq i \leq k$. As a result we have

$$\|x_n - x\|^2 = \sum_{i=1}^{k} |\langle x_n - x, u_i\rangle|^2 \to 0 \quad \text{as } n \to \infty.$$

Problem 8.19: Consider the Hilbert space $\mathcal{H}_1 := \mathcal{H} \oplus \mathbb{R}$, and define $K := \{(x,a) \in \mathcal{H}_1 : f(x) \leq a\} \subset \mathcal{H}_1$. Since f is convex, for any $(x,a),(y,b) \in K$ we have $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y) \leq ta+(1-t)b$ i.e., $t(x,a)+(1-t)(y,b) \in K$. Therefore K is convex.

Secondly, let $K \ni (x_n, a_n) \to (x,a) \in \mathcal{H}_1$ strongly. Then, since f is strongly lower-semicontinuous, $f(x) \leq \liminf f(x_n) \leq \liminf a_n = a$, which implies that $(x,a) \in K$ and thus K is strongly closed. Therefore by Mazur's lemma, K is weakly closed.

Now let $\mathcal{H} \ni x_n \rightharpoonup x \in \mathcal{H}$. Construct a subsequence $\{x_{n_k}\}$ such that $f(x_{n_k}) \leq m_k + \frac{1}{k}$, where $m_{k+1} = \inf_{n \geq n_k} f(x_n)$, and thus $(x_{n_k}, m_k + 1/k) \in K$. Since $\lim_{k\to\infty} m_k = \liminf f(x_n)$, and $x_n \rightharpoonup x$, we conclude that $K \ni (x_{n_k}, m_k + 1/k) \rightharpoonup (x, \liminf f(x_n))$. But K is weakly closed, therefore $(x, \liminf f(x_n)) \in K$. As a result

$$f(x) \leq \liminf f(x_n).$$