MAT202
Midterm Exam (Due in class on February 15, 2019).

Problem 1.

Let $X = C[0,1]$ and define $D = \{f \in X : df/dx \in X\}$. Let $A : X \to X$ be the differentiation operator, i.e. $Af = df/dx$. Prove that A is discontinuous but closed.

Problem 2.

Let $\{T_n\}$ be a sequence of bounded linear operators between Banach spaces X and Y such that for every $x \in X$ the limit $Tx := \lim_{n\to\infty} T_n x$ exists. Prove that T is also a bounded linear operator between X and Y and

$$||T|| \leq \liminf ||T_n||.$$

Problem 3.

Prove that a convex closed set in a Banach space is perfectly convex. Give an example of a convex set which is not perfectly convex.

Problem 4.

Prove that the collection of all monomials $\{x^n,\ n = 0, 1, 2, \ldots\}$ is not a basis of $L^2[0,1]$ in the sense of Definition 9.5.2.

Problem 5.

Let $l^\infty(R)$ be the space of bounded one-sided infinite real sequences $\{x_n\}_1^\infty$ with the supremum (uniform) norm. Prove that there exists a bounded linear functional LIM on $l^\infty(R)$ such that

$$(i) \sup_n x_n \geq LIM\{x_n\} \geq \inf x_n.$$

$$(ii) LIM\{x_n\} = \lim_n x_n$$

if the limit $\lim x_n$ exists.

Problem 6.

Let T be a bounded invertible linear operator acting on a Banach space X such that there exists $const > 0$ that

$$||T^n x|| \leq const||x||, \quad x \in X, \quad n = 0, 1, -1, 2, -2, \ldots$$

Prove that there exists another norm on X equivalent to the original one such that T is an isometry with respect to the new norm.