3. Exercises

for some integers \( n \) and \( m \).

[Hint: If the boundary of the parallelogram contains no zeros or poles, simply integrate \( z f'(z)/f(z) \) over that boundary, and observe that the integral of \( f'(z)/f(z) \) over a side is an integer multiple of \( 2\pi i \). If there are zeros or poles on the side of the parallelogram, translate it by a small amount to reduce the problem to the first case.]

3. In contrast with the result in Lemma 1.5, prove that the series

\[
\sum_{n+m\tau \in \Lambda} \frac{1}{|n+m\tau|^2} \quad \text{where } \tau \in \mathbb{H}
\]

does not converge. In fact, show that

\[
\sum_{1 \leq n^2 + m^2 \leq R^2} \frac{1}{(n^2 + m^2)^2} = 2\pi \log R + O(1) \quad \text{as } R \to \infty.
\]

4. By rearranging the series

\[
\frac{1}{x^2} + \sum_{\omega \in \Lambda} \left[ \frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \right],
\]

show directly, without differentiation, that \( \rho(z + \omega) = \rho(z) \) whenever \( \omega \in \Lambda \).

[Hint: For \( R \) sufficiently large, note that \( \rho(z) = \rho^R(z) + O(1/R) \), where \( \rho^R(z) = z^{-2} + \sum_{0 < |\omega| < R} ((z + \omega)^{-2} - \omega^{-2}) \). Next, observe that both \( \rho^R(z + 1) - \rho^R(z) \) and \( \rho^R(z + \tau_R) - \rho^R(z) \) are \( O(\sum_{R - \epsilon < |\omega| < R + \epsilon} |\omega|^{-2}) = O(1/R) \).]

5. Let \( \sigma(z) \) be the canonical product

\[
\sigma(z) = z \prod_{j=1}^{\infty} E_2(z/\tau_j),
\]

where \( \tau_j \) is an enumeration of the periods \( \{n + m\tau \} \) with \( (n, m) \neq (0, 0) \), and \( E_2(z) = (1 - z)e^{z + z^2/2} \).

(a) Show that \( \sigma(z) \) is an entire function of order 2 that has simple zeros at all the periods \( n + m\tau \), and vanishes nowhere else.

(b) Show that

\[
\frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{(n,m) \neq (0,0)} \left[ \frac{1}{z - n - m\tau} + \frac{1}{n + m\tau} + \frac{z}{(n + m\tau)^2} \right],
\]

and that this series converges whenever \( z \) is not a lattice point.
(c) Let $L(z) = -\sigma'(z)/\sigma(z)$. Then

$$L'(z) = \frac{(\sigma'(z))^3 - \sigma(z)\sigma''(z)}{(\sigma(z))^2} = \rho(z).$$

6. Prove that $\rho''$ is a quadratic polynomial in $\rho$.

7. Setting $\tau = 1/2$ in the expression

$$\sum_{m=1}^{\infty} \frac{1}{(m + \tau)^2} = \frac{\pi^2}{\sin^2(\pi \tau)},$$

deduce that

$$\sum_{m \geq 1, m \text{ odd}} \frac{1}{m^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{m \geq 1} \frac{1}{m^2} = \frac{\pi^2}{6} = \zeta(2).$$

Similarly, using $\sum 1/(m + \tau)^4$ deduce that

$$\sum_{m \geq 1, m \text{ odd}} \frac{1}{m^4} = \frac{\pi^4}{96} \quad \text{and} \quad \sum_{m \geq 1} \frac{1}{m^4} = \frac{\pi^4}{90} = \zeta(4).$$

These results were already obtained using Fourier series in the exercises at the end of Chapters 2 and 3 in Book I.

8. Let

$$E_4(\tau) = \sum_{(n,m) \neq (0,0)} \frac{1}{(n + m\tau)^4}$$

be the Eisenstein series of order 4.

(a) Show that $E_4(\tau) \to \pi^4/45$ as $\text{Im}(\tau) \to \infty$.

(b) More precisely,

$$\left| E_4(\tau) - \frac{\pi^4}{45} \right| \leq c e^{-2\pi t} \quad \text{if } \tau = x + it \text{ and } t \geq 1.$$  

(c) Deduce that

$$\left| E_4(\tau) - \tau^{-1}\pi^4/45 \right| \leq c t^{-1} e^{-2\pi/t} \quad \text{if } \tau = it \text{ and } 0 < t \leq 1.$$