Midterm 2 Solutions for S2018 MAT-21A

Problem 1

Find the derivative of the function $\frac{x^3-e^{2x}}{3e^{-x}-x}$

Quotient rule:

$$\frac{(3x^2-2e^{2x})(3e^{-x}-x)-(x^3-e^{2x})(-3e^{-x}-1)}{(3e^{-x}-x)^2}$$

Problem 2

Find the derivative of the function $\frac{\sin(x)}{x}+\frac{x}{\sin(x)}$

$\frac{\cos(x)x-\sin(x)}{x^2}+\frac{\sin(x)-x\cos(x)}{\sin^2(x)}$

Problem 3

Find the second derivative of $e^{x^2}-3x^2$

First derivative: $2xe^{x^2}-6x$

Second derivative: $2e^{x^2}+4x^2e^{x^2}-6$

Problem 4

Find $\frac{dy}{dx}$ at $(0,\pi)$ where $x^2\cos^2(y)-\sin(y)=0$

$2x\cos^2(y) - 2x^2\cos(y)\sin(y)y' - \cos(y)y' = 0$

$y'(2x^2\cos(y)\sin(y) + \cos(y)) = 2x\cos^2(y)$

$y' = \frac{2x\cos^2(y)}{2x^2\cos(y)\sin(y)+\cos(y)}$

At $(0, \pi)$: $y' = \frac{0}{0+1} = 0$

Problem 5

Verify the following curves meet orthogonally

$x^2 + y^2 = 4$ and $x^2 = 3y^2$

Curve 1: $2x + 2yy' = 0$ so $y' = -\frac{x}{y}$

Curve 2: $2x = 6yy'$ so $y' = \frac{x}{3y}$

Curves meet when both are satisfied so we require $4y^2 = 4$ or $x = \pm\sqrt{3}$ and $y = \pm 1$, so $x = \pm y\sqrt{3}$

Evaluate slopes

- Curve 1: $y' = \mp\sqrt{3}$
- Curve 2: $y' = \pm\frac{1}{\sqrt{3}}$

They are orthogonal because the slopes are the negative reciprocal of each other.

Problem 6

Find the derivative of $(x+2)^{x^2+2}$

$y = (x+2)^{x^2+2}$

$\ln y = (x^2+2)\ln(x+2)$

$\frac{y'}{y} = 2x\ln(x+2) + \frac{x^2+2}{x+2}$

$y' = 2x\ln(x+2)(x+2)^{x^2+2} + (x^2+2)(x+2)^{x^2+1}$

Problem 7

Differentiate

$$\tan^{-1}(e^{-x} + \log(x))$$

where $\log(x)$ is the natural logarithm

$$\frac{-e^{-x} + \frac{1}{x}}{(e^{-x} + \log(x))^2 + 1}$$

Problem 8

If the original 24 m edge length x of a cube decreases at the rate of 5 m/min, when $x = 4$ at what rate does the cube's volume change?

Given information: Want $\frac{dV}{dt}$ when $x = 4$ and $\frac{dx}{dt} = -5$

Cube volume: $V = x^3$

Implicit differentiation: $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

Plug in given values: $\frac{dV}{dt} = 3 * 16 * (-5) = -240 \frac{\text{m}}{\text{min}}$

Problem 9

Find the linearization of $f(x) = 3^x$ at $x = 1$

$f'(x) = \log(3)3^x$

$f(1) = 3$

$f'(1) = 3\log(3)$

$f(x) \approx f(a) + f'(a)(x - a) = 3 + 3\log(3)(x - 1)$