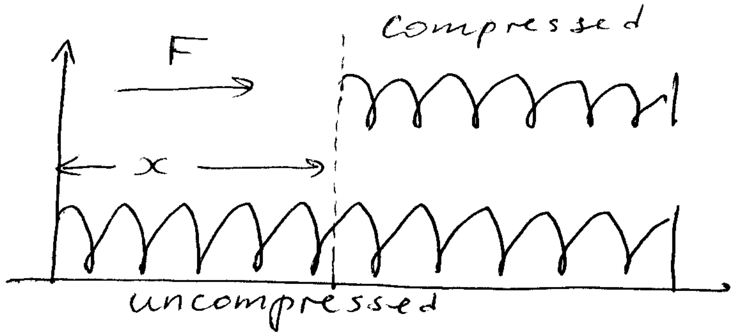
5. It takes a force of 21,714 lb. to compress a coil spring assembly on a New York City Transit Authority subway car from its free height of 8 in. to its fully compressed height of 5 in.

a) What is the assembly's force constant (a.k.a. spring constant)?

b) How much work does it take to compress the assembly the first half inch? the second half inch?

a)



Hooke's Law

$$F = kx = k(8-5) = 3k$$

$$21{,}714 = 3k \Rightarrow k = 7{,}238$$

b)

$$W_1 = \int_0^{\frac{1}{2}} F(x)\,dx \quad \leftarrow \text{first half inch}$$

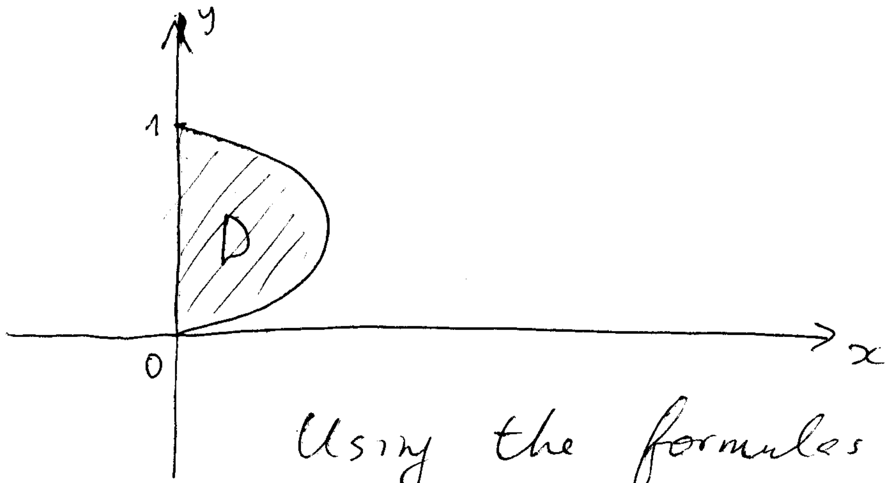
$$W_2 = \int_{\frac{1}{2}}^{1} F(x)\,dx \quad \leftarrow \text{second half inch}$$

In our case, $F(x) = kx$. Thus

$$W_1 = \int_0^{\frac{1}{2}} kx\,dx = k\frac{x^2}{2}\Big|_0^{\frac{1}{2}} = \frac{k}{8} = \frac{7{,}238}{8} = 904\tfrac{3}{4} \text{ lb-in}$$

$$W_2 = \int_{\frac{1}{2}}^{1} kx\,dx = k\frac{x^2}{2}\Big|_{\frac{1}{2}}^{1} = \frac{k}{2} - \frac{k}{8} = \frac{3}{8}k = \frac{3}{8}\cdot 7{,}238 \text{ lb-in}$$

6. Find the center of mass of a thin plate of constant density δ covering the region bounded by the y-axis and the curve $x = y - y^3$, $0 \le y \le 1$.



Using the formulas on p. 404,

$$A(D) = \int_0^1 (y - y^3)\,dy = \frac{y^2}{2} - \frac{y^4}{4}\Big|_0^1 = \frac{1}{4}$$

$$\bar{x} = \frac{\int_0^1 y\,(y-y^3)\,dy}{A(D)} = 4\int_0^1 y(y-y^3)\,dy = 4\int_0^1 (y^2 - y^4)\,dy$$

$$= 4\left(\frac{y^3}{3} - \frac{y^5}{5}\right)\Big|_0^1 = 4\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8}{15}$$

$$\bar{y} = \frac{\int_0^1 \frac{1}{2}(y-y^3)^2\,dy}{A(D)} = 2\int_0^1 (y-y^3)^2\,dy$$

$$= 2\int_0^1 \left(y^2 - 2y^4 + y^6\right)dy = 2\left(\frac{y^3}{3} - \frac{2}{5}y^5 + \frac{y^7}{7}\right)\Big|_0^1$$

$$= 2\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) = 2\left(\frac{35}{105} - \frac{42}{105} + \frac{15}{105}\right) = \frac{16}{105}$$

7. Solve the differential equation

$$2\sqrt{xy}\frac{dy}{dx} = 1, \quad x, y > 0.$$

This is a Separable Differential Equation (see pp 428-430). We write:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{xy}}$$

$$\sqrt{y}\, dy = \frac{1}{2} \frac{dx}{\sqrt{x}}$$

$$\int y^{\frac{1}{2}} dy = \frac{1}{2} \int \frac{dx}{x^{\frac{1}{2}}}$$

$$\frac{2}{3} y^{\frac{3}{2}} = x^{\frac{1}{2}} + C$$

$$y^{\frac{3}{2}} = \frac{3}{2} x^{\frac{1}{2}} + C_1$$

$$y = \left(\frac{3}{2} x^{\frac{1}{2}} + C_1\right)^{\frac{2}{3}}, \qquad C_1 \geq 0.$$

8. Suppose the amount of oil pumped from one of the canyon wells in Whittier, California decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present value?

Look at the Exponential Change section
(p. 428).

We have

$$y(t) = y(0)\, e^{kt}, \text{ where}$$

$$k = \ln(0.9), \text{ so } y(t) = y(0)\,(0.9)^t$$

$$y(t) = \tfrac{1}{5}\, y(0) \Rightarrow$$

$$\Rightarrow \tfrac{1}{5} = (0.9)^t = e^{\ln 0.9\, t}$$

$$\Rightarrow \ln 0.2 = t \ln 0.9$$

$$\Rightarrow t = \frac{\ln 0.2}{\ln 0.9} = \frac{\ln 5}{\ln 10 - \ln 9}$$