1. Does the following sequence converge or diverge? If it converges, find its limit. Explain your answer.

(a)

$$a_n = \sqrt[n]{4^n n^4}$$

Observe that $a_n = 4 n^{\frac{4}{n}} = 4(\sqrt[n]{n})^4$.

Theorem 5 part (2) on p. 738 states that

$\sqrt[n]{n} \to 1$ as $n \to \infty$. To see it write

$\sqrt[n]{n} = n^{\frac{1}{n}} = e^{\frac{\ln n}{n}}$. Now $\lim\limits_{n\to\infty} \frac{\ln n}{n} =$

$\lim\limits_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$ by L'Hôpital Rule (pp. 736-738)

Thus $n^{\frac{1}{n}} \to e^0 = 1$ and $a_n = 4(n^{\frac{1}{n}})^4 \to 4 \cdot 1^4 = 4$

Therefore, $\boxed{\lim a_n = 4}$

(b)

$$a_n = \frac{2010^n}{n!}$$

Theorem 5 part (6) on p. 738 states that

$\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0$ (any x)

Thus, $\boxed{\lim\limits_{n\to\infty} \frac{2010^n}{n!} = 0}$

To see this, write

$$a_n = \frac{\overbrace{2010 \times 2010 \times 2010 \times \cdots \times 2010 \times 2010 \times \cdots 2010}^{\longleftarrow\ n \text{ times} \longrightarrow}}{1 \times 2 \times 3 \times \cdots \times 2010 \times 2011 \times \cdots n}$$

$$\leq \frac{2010^{2010}}{2010!} \cdot \frac{2010}{n} \quad \text{for } n \geq 2010$$

Since $\frac{2010}{n} \to 0$ we have that $a_n \to 0$ by the Sandwich Theorem (p 735)

2. Compute the following infinite sum. Explain your answer.

(a)

$$\sum_{n=1}^{\infty} \frac{1}{2n(n+1)}$$

<u>Remark</u> Compare with the Example 5 on p 749 $\left(\sum_{n=1}^{\infty} \frac{1}{n(n+1)}\right)$. We also did a similar example in class.

$\triangleright$

$$S_n = \sum_{k=1}^{n} \frac{1}{2k(k+1)} = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{n+1}\right) \to \frac{1}{2} \text{ as } n \to \infty. \quad \text{Thus } \boxed{\sum_{n=1}^{\infty} \frac{1}{2n(n+1)} = \frac{1}{2}}$$

(b)

This is a geometric series! (pp 747–748)

$$\sum_{n=2}^{\infty}\left(-\frac{4}{5}\right)^n$$

$$\left(-\frac{4}{5}\right)^2 + \left(-\frac{4}{5}\right)^3 + \left(-\frac{4}{5}\right)^4 + \left(-\frac{4}{5}\right)^5 + \cdots$$

$$= \left(-\frac{4}{5}\right)^2 \left(1 + \left(-\frac{4}{5}\right) + \left(-\frac{4}{5}\right)^2 + \left(-\frac{4}{5}\right)^3 + \cdots\right)$$

$$= \left(-\frac{4}{5}\right)^2 \underbrace{\sum_{n=0}^{\infty}\left(-\frac{4}{5}\right)^n}_{\text{geometric series}} = \left(-\frac{4}{5}\right)^2 \frac{1}{1-\left(-\frac{4}{5}\right)}$$

$$= \frac{16}{25} \frac{1}{1+\frac{4}{5}} = \frac{16}{25} \frac{5}{9} = \boxed{\frac{16}{45}}$$

3. Does the following infinite series converge or diverge? Explain your answer.

(a)

$$\sum_{n=1}^{\infty} \frac{n^2+4n-3}{n^4+4n^2-3}$$

Answer: Converges.

$\triangleright$ Let $a_n = \dfrac{n^2+4n-3}{n^4+4n^2-3}$ and $b_n = \dfrac{1}{n^2}$

Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n^2+4n-3}{n^4+4n^2-3} \Big/ \frac{1}{n^2} \right)$

$\displaystyle = \lim_{n\to\infty} \frac{n^4+4n^3-3n^2}{n^4+4n^2-3} = \lim_{n\to\infty} \frac{1+\frac{4}{n}-\frac{3}{n^2}}{1+\frac{4}{n^2}-\frac{3}{n^4}}$

$= 1$. Since $\sum b_n = \sum \frac{1}{n^2}$ converges (P-Series, Example 3 on p. 758), by the Limit Comparison Test (Thm. 11 p. 762), $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(b)

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{\ln(2n)}$$

$\triangleright$ Answer: Diverges

$\displaystyle a_n = \frac{\ln n}{\ln(2n)} = \frac{\ln n}{\ln 2 + \ln n} = \frac{1}{\frac{\ln 2}{\ln n} + 1}$

Thus $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\frac{\ln 2}{\ln n}+1}$

$\displaystyle = \frac{1}{0+1} = 1$ since $\ln n \nearrow +\infty$

Thus $\sum a_n$ diverges by the n-th Term Test for Divergence (see Thm. 7 on p. 750), since $\displaystyle\lim_{n\to\infty} a_n = 1 \neq 0$.

4. Determine whether the following infinite series absolutely converges, conditionally converges, or diverges. Give reasons for your answer.

(a)

$$\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+1}-\sqrt{n})$$

<u>Remark</u> This problem is from the Theoretical Problem Set #3 (problem 11.6.39).

▷ $\sqrt{n+1}-\sqrt{n} = (\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})/(\sqrt{n+1}+\sqrt{n}) = \dfrac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}}$

Thus, the infinite sum is $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n+1}+\sqrt{n}}$

It <u>converges</u> by the <u>Alternating Series Test</u> (p. 771)

However, $\displaystyle\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\sqrt{n+1}-\sqrt{n}$ diverges since

$S_N = \displaystyle\sum_{n=1}^{N}|a_n| = \sqrt{N+1}-\sqrt{1} \to \infty$ (use telescoping property).

Thus $\displaystyle\sum_{n=1}^{\infty} a_n$ conditionally converges.

(b)

$$\frac{1}{1}-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}-\frac{1}{9}+\frac{1}{10}-\frac{1}{11}-\frac{1}{12}+\cdots$$

▷ The series diverges. Clearly, it does not converge absolutely since $\displaystyle\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{n}$ which is the <u>Harmonic Series</u> (Example 1 on pp 756-757)

To see that $\sum a_n$ diverges, write

$$S_{3N} = \sum_{n=0}^{N-1}\left(\frac{1}{3n+1}-\frac{1}{3n+2}-\frac{1}{3n+3}\right) = \sum_{n=0}^{N-1} -\frac{9n^2+6n-1}{(3n+1)(3n+2)(3n+3)}.$$

Use the <u>Limit Comparison Test</u> to compare

$\displaystyle\sum_{n=0}^{\infty}\frac{9n^2+6n-1}{(3n+1)(3n+2)(3n+3)}$ with the <u>Harmonic series</u>

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ since $\displaystyle\lim_{n\to\infty}\frac{9n^2+6n-1}{(3n+1)(3n+2)(3n+3)}\Big/\frac{1}{n} = \frac{9}{27} = \frac{1}{3} > 0.$

5. Find the radius and interval of convergence of

$$\sum_{n=2}^{\infty} \frac{(2x)^n}{n\ln(n)}$$

For what values of x does the series converge absolutely? For what values of x does the series converge conditionally?

To find the Radius of Convergence we check for absolute convergence by the Ratio Test (Thm. 12 p. 766).

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|2x|^{n+1}}{(n+1)\ln(n+1)} \Big/ \frac{|2x|^n}{n\ln n}$$

$$= \lim_{n\to\infty} |2x| \frac{n}{(n+1)} \cdot \frac{\ln n}{\ln(n+1)} = |2x| \cdot 1 \cdot 1 = |2x|$$

since $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 0$ and

$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = 1$ by L'Hôpital.

Thus, the series converges absolutely when $|2x| < 1 \iff |x| < \frac{1}{2}$, so $R = \frac{1}{2}$.

Therefore, the series diverges for $|x| > \frac{1}{2}$

Endpoints: $x = \pm\frac{1}{2}$. When $x = -\frac{1}{2}$, we have

$\sum_{n=2}^{\infty} \frac{(2\cdot(-\frac{1}{2}))^n}{n\ln n} = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n\ln n}$ and it converges

by the Alternating Series Test (Thm. 14, p. 771)

since $u_n = \frac{1}{n\ln n} > 0$, $u_n > u_{n+1}$ for all n & $\lim_{n\to\infty} u_n = 0$

When $x = \frac{1}{2}$ we obtain $\sum_{n=1}^{\infty} \frac{1}{n\ln n}$. It diverges by the Integral Test (Thm 9 p. 757).

6. (**Bonus Problem**)

(a) Show that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers then $\sum_{n=1}^{\infty} a_n^2$ converges.

(b) Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} a_n^2$ diverges.

Hint: Part a) suggests that for an example in part b) the terms of the series $\sum_{n=1}^{\infty} a_n$ cannot be all positive (or all negative).

Remark Part (a) is from Theoretical Problem Set #2 (problem 11.4.40)

a) If $\sum_{n=1}^{\infty} a_n$ converges then $a_n \to 0$ as $n \to \infty$ (n^{th} Term Test, Theorem 7 p. 750). Thus $0 \leq a_n \leq 1$ for all $n \geq N$. Then

$0 \leq a_n^2 \leq a_n$ for all $n \geq N$

By Comparison Test (Theorem 10, p. 761) the series $\sum_{n=1}^{\infty} a_n^2$ converges since $\sum_{n=1}^{\infty} a_n$ converges

(b) Let $a_n = \frac{(-1)^n}{\sqrt{n}}$

Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges by the Alternating Series Test (Theorem 14, p. 771)

since $u_n = \frac{1}{\sqrt{n}} \geq 0$, $u_{n+1} \leq u_n$ for all $n \geq 1$

and $u_n \to 0$ as $n \to \infty$.

However $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

(Harmonic Series, see Example 1 on pp. 756-757)