1. Represent the function $f(x) = 10\ln(1+3x)$ as a power series. Find its radius of convergence and its interval of convergence.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x|<1$$

Thus, $$\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-1)^n (3x)^n = \sum_{n=0}^{\infty} (-1)^n 3^n x^n$$
for $|x| < \frac{1}{3}$

Then
$$\int \frac{1}{1+3x}\,dx = \sum_{n=0}^{\infty} (-1)^n 3^n \int x^n\,dx$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n+1} x^{n+1} + C$$

In particular
$$\frac{1}{3}\ln(1+3x) = \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n+1} x^{n+1} \quad \text{for } |x| < \frac{1}{3}$$
($C=0$ since functions agree at $x=0$)

Therefore,
$$\boxed{\begin{aligned} 10\ln(1+3x) &= 30\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n+1} x^{n+1} \\ &= \sum_{n=0}^{\infty} (-1)^n\, 30 \cdot \frac{3^n}{n+1} x^{n+1} \quad \text{for } |x| < \tfrac{1}{3} \end{aligned}}$$

The power series converges at $x=\frac{1}{3}$ (alternating series test) and diverges at $x=-\frac{1}{3}$ (Harmonic Series). Thus, $R=\frac{1}{3}$ and the interval of convergence $=(-\frac{1}{3}, \frac{1}{3}]$.

2. Find the Maclaurin polynomial of degree 5 for

$$F(x) = \int_0^x e^{-\frac{t^2}{2}}\,dt.$$

Use this polynomial to estimate

$$\int_0^1 e^{-\frac{t^2}{2}}\,dt.$$

Give an upper bound for your error of your estimate.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. Thus, $e^{-\frac{t^2}{2}} = \sum_{n=0}^{\infty} \left(-\frac{t^2}{2}\right)^n / n!$

$= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{2^n n!}$ for all t.

Therefore, $F(x) = \int_0^x e^{-\frac{t^2}{2}}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n n!} \times$

$\int_0^x t^{2n}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n n!} \frac{x^{2n+1}}{2n+1}.$

The Maclaurin polynomial of degree 5 is

$$\boxed{P_5(x) = x - \frac{1}{2\cdot 1}\frac{x^3}{3} + \frac{1}{2^2\cdot 2}\frac{x^5}{5} = x - \frac{x^3}{6} + \frac{x^5}{40}}$$

Thus, we can estimate $\int_0^1 e^{-t^2/2}\,dt$ as

$P_5(1) = 1 - \frac{1}{6} + \frac{1}{40} = \frac{120-20+3}{120} = \frac{103}{120}.$

Since $\int_0^1 e^{-\frac{t^2}{2}}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n n!(2n+1)}$

is an alternating series satisfying the conditions of the Alternating Series Test, we have

$$\left| \int_0^1 e^{-\frac{t^2}{2}}\,dt - \frac{103}{120} \right| \leq \frac{1}{2^3\, 3!\,(2\cdot 3+1)} = \frac{1}{336}$$

3. (a) Write down the Maclaurin series for $f(x) = \cos(x^4/4)$.
(b) Compute the 10^{th} derivative of $\cos(x^4/4)$ at $x = 0$.

▷

a) Since $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, we have

$$\cos(x^4/4) = \sum_{n=0}^{\infty} (-1)^n (x^4/4)^{2n} / (2n)!$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{1}{4^{2n}(2n)!} x^{8n}}$$

$$= 1 - \frac{x^8}{32} + \frac{1}{4^4\, 4!} x^{16} - \dots$$

b) The 10^{th} derivative of $\cos\left(\frac{x^4}{4}\right)$ at $x = 0$ equals zero since the coefficient in front of x^{10} is zero in the Maclaurin series (we use the fact that the coefficient in front of x^{10} equals $\frac{f^{(10)}(0)}{10!}$).

4. (a) Find the parametric equations for the line of intersection of the planes $x+y+z=3$ and $2x-3y=6$.
(b) Find the distance between the point $P(0,0,1)$ and the line of intersection.

▷ a) A normal to the first plane can be taken to be $n_1 = \langle 1,1,1 \rangle$. A normal to the second plane can be taken to be $\langle 2,-3,0 \rangle$. Then v can be chosen to be $n_1 \times n_2$.

$$v = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & -3 & 0 \end{vmatrix} = 3i + 2j - 5k$$

To find a point $Q(x_0, y_0, z_0)$ on the intersection line we have to find one solution of

$$\begin{cases} x+y+z=3 \\ 2x-3y=6 \end{cases}$$

Choosing $x=0$, we get $y=-2$, $z=5$

Thus, $Q(0,-2,5)$ is on the line.

Therefore, $\boxed{\begin{cases} x = 0+3t \\ y = -2+2t \\ z = 5-5t \end{cases}}$ are parametric equations of the intersection line.

b) We have $\dfrac{v}{|v|} = \dfrac{\langle 3,2,-5 \rangle}{\sqrt{3^2+2^2+5^2}} = \dfrac{\langle 3,2,-5 \rangle}{\sqrt{38}}$

Then $d = \left| \overrightarrow{PQ} \times \dfrac{v}{|v|} \right| = \dfrac{|\overrightarrow{PQ} \times v|}{|v|}$

We have $\overrightarrow{PQ} \times v = \begin{vmatrix} i & j & k \\ 0 & -2 & 4 \\ 3 & 2 & -5 \end{vmatrix} = 2i + 12j + 6k$,

so $|\overrightarrow{PQ} \times v| = \sqrt{4+144+36} = \sqrt{184}$ and $d = \dfrac{\sqrt{184}}{\sqrt{38}}$

5. (a) Find the equation of the plane that passes through the three points $A(1,0,0)$, $B(0,2,0)$, and $C(0,0,3)$.

(b) Find the distance between the origin and this plane.

▷ a) $\overrightarrow{AB} = \langle -1, 2, 0 \rangle$, $\overrightarrow{AC} = \langle -1, 0, 3 \rangle$

Then $n = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6i + 3j + 2k$

Therefore, we can write an equation of the plane as

$6(x-1) + 3(y-0) + 2(z-0) = 0$

$\Leftrightarrow \boxed{6x + 3y + 2z = 6}$

b) $\dfrac{n}{|n|} = \dfrac{\langle 6, 3, 2 \rangle}{\sqrt{36+9+4}} = \langle \frac{6}{7}, \frac{3}{7}, \frac{2}{7} \rangle$

$d = \left| \overrightarrow{OA} \cdot \dfrac{n}{|n|} \right| = \left| \langle 1,0,0 \rangle \cdot \langle \frac{6}{7}, \frac{3}{7}, \frac{2}{7} \rangle \right| = \frac{6}{7}$

Therefore, $\boxed{d = \frac{6}{7}}$

6. (**Bonus Problem**) Given nonzero vectors u,v, and w, use dot product and cross product notation, as appropriate, to describe the following.

(a) The vector projection of u onto v.

(b) A vector orthogonal to u and v.

(c) A vector orthogonal to $u \times v$ and w (you can assume that u is not orthogonal to w).

(d) The area of the parallelogram determined by u and w.

a) $\mathrm{proj}_V\, u = \left(\frac{u \cdot V}{|V|^2}\right) V$

b) $u \times V$

c) $(u \times V) \times W$ This would give you 2 points out of 2.5 maximum. To get the maximum 2.5 points, you were expected to note that any vector orthogonal to $u \times v$ can be written as $au + bv$. To find a and b you can use

$$(au + bv) \cdot W = 0 \iff a(u \cdot W) + b(v \cdot w) = 0$$

so you can choose $a = 1$, $b = -\frac{v \cdot w}{u \cdot w}$

d) $|u \times W|$