6.6 (2)

Let $S = \{v_1, v_2, v_3, v_4, v_5\}$

$v_1 = (1,1,2,1)$ $v_2 = (1,0,-3,1)$

$v_3 = (0,1,1,2)$ $v_4 = (0,0,1,1)$

$v_5 = (1,0,0,1)$

$V = \text{Span } S$

$$A = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \\ \ \end{matrix}$$

V is row space of A + since row space A = row space B

$\Rightarrow$ $\boxed{\text{a basis for } V = \{[1\,0\,0\,0], [0\,1\,0\,0], [0\,0\,1\,0], [0\,0\,0\,1]\}}$

(5) $$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 9 & -1 \\ -3 & 8 & 3 \\ -2 & 3 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \ \\ \ \end{matrix}$$

a) basis for row space of $A = \{[1\,0\,-1], [0\,1\,0]\}$

b) or, $RREF(A^T) = \begin{bmatrix} 1 & 0 & -5 & -3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow$ basis row space A $= \{[1\,2\,-1], [1\,9\,-1]\}$

$\uparrow \uparrow$

(9) $$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 3 & 2 \\ 0 & -7 & 8 \end{bmatrix} \qquad RREF(A) = \begin{bmatrix} 1 & 0 & 19/7 \\ 0 & 1 & -8/7 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \ \end{matrix}$$

$$RREF(A^T) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \ \end{matrix}$$

a) Look at RREF(A)

$\Rightarrow \{[1\,0\,19/7], [0\,1\,-8/7]\}$ = basis for row space of $\underline{A}$

b) Look at $RREF(A^T)$

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ = basis for col space of $\underline{\underline{A}}$

c) also $\{[1\,0\,2], [0\,1\,-1]\}$ = basis for row space of $\underline{\underline{A^T}}$

d) also $\left\{ \begin{bmatrix} 1 \\ 0 \\ 19/7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -8/7 \end{bmatrix} \right\}$ = basis for col space of $\underline{\underline{A^T}}$

note relationships between vectors and row/col. spaces of A, A^T

6.6 (15) $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ rank $A = 3$

A represents co-efficients of homogeneous system of 3 eqns, 3 unks,
as does RREF(A) $= I_3$
$\Rightarrow$ no variable can be arbitrarily chosen
$\Rightarrow$ Nullity $= 0$
Rank + Nullity $= 3 + 0 = 3 = n$

(20) A is 5×3 Matrix
Show rows of A are LD

A has 5 rows $\Rightarrow$
5 rows span a row space of dim $\leq$ rank $A = 3$
$\Rightarrow$ rows must be LD

(23) $A = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 2 & 3 \\ 0 & 8 & 0 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A) = 2 < n$

by Th 6.13 A is <u>singular</u>

(32) $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 5 & -4 & 3 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(A) = 3 = n$

by Cor 6.5 <u>$\exists$ only the trivial sol'n</u>

6.6 (T2) Prove Cor 6.3

Let A be $n \times n$ matrix

$Ax = b$ has unique sol'n for every $b\,(n\times 1) \iff \text{rank } A = n$

$\Leftarrow$ Let rank $A = n$

Th 6.13 $\Rightarrow$ A non-singular

$\Rightarrow x = A^{-1}b$ is a sol'n to $Ax = b$

Suppose x_1 & x_2 are both sol'n's to $Ax = b$

$\Rightarrow Ax_1 = Ax_2$

$\Rightarrow A^{-1}Ax_1 = A^{-1}Ax_2$

$\Rightarrow \quad x_1 = x_2 \quad \Rightarrow \quad Ax = b$ has unique sol'n, as required ■

$\Rightarrow$ Suppose $Ax = b$ has unique sol'n for every $b\,(n\times 1)$.

$\Rightarrow Ax = e_1, Ax = e_2, \ldots, Ax = e_n$ ($e_1, \ldots, e_n$ columns of I_n)

have unique sol'n's $x_1, x_2, \ldots, x_n$

Let B be the matrix whose j^{th} col is x_j

$\Rightarrow$ we can write the linear system as

$AB = I_n$

$\Rightarrow B = A^{-1} \Rightarrow A$ non-singular

$\Rightarrow$ (by Th 6.13) $\text{rank}(A) = n$, as required ■

(T7) A an $m \times n$ matrix

Show $Ax = b$ has sol'n for all $b\,(m\times 1) \iff \text{rank}(A) = m$

$\Rightarrow$ $Ax = b$ has sol'n $\forall\, b\,(m\times 1)$

$\Rightarrow$ col's of A span $\mathbb{R}^m$

$\Rightarrow$ a subset of columns of A is a basis for $\mathbb{R}^m$ & $\text{rank}(A) = m$, as req'd

$\Leftarrow$ Suppose $\text{rank}(A) = m$

$\Rightarrow$ col. rank$(A) = m$

$\Rightarrow$ m col's of A are a basis for $\mathbb{R}^m$

$\Rightarrow$ the col's of A span $\mathbb{R}^m$

Now, $b \in \mathbb{R}^m \Rightarrow b$ consists of a linear comb. of col's of A

$\Rightarrow Ax = b$ has a sol'n $\forall\, b\,(m\times 1)$, as req'd ■

6.7 ① $V = \mathbb{R}^2$

$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \ v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 3 \\ -2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow [v]_S = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

④ $V = P_2$

$S = \{ t^2 - t + 1, \ t + 1, \ t^2 + 1 \}; \ v = 4t^2 - 2t + 3$

$c_1(t^2 - t + 1) + c_2(t+1) + c_3(t^2+1) = 4t^2 - 2t + 3$

$(c_1 + c_3)t^2 + (-c_1 + c_2)t + (c_1 + c_2 + c_3) = 4t^2 - 2t + 3$

$$\left.\begin{aligned} c_1 \quad\quad + c_3 &= 4 \\ -c_1 + c_2 \quad\quad &= -2 \\ c_1 + c_2 + c_3 &= 3 \end{aligned}\right\} \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ -1 & 1 & 0 & -2 \\ 1 & 1 & 1 & 3 \end{array}\right] \xrightarrow[\text{ops}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\Rightarrow \boxed{[v]_S = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}$

⑤ $V = M_{22}$

$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \ v = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = 1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

So $\begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

$\Rightarrow [v]_S = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$

6.7 (13) $S = \{(1,2),(0,1)\}$, $T = \{(1,1),(2,3)\}$ bases for $\mathbb{R}^2$; $v = (1,5)$, $w = (5,4)$

a) Find $[v]_T$, $[w]_T$

$c_1(1,1) + c_2(2,3) = (1,5)$

$c_1 + 2c_2 = 1$

$c_1 + 3c_2 = 5$

$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 3 & 5 \end{array}\right] \xrightarrow[\text{ops}]{\text{row}} \left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & 4 \end{array}\right]$

$c_3(1,1) + c_4(2,3) = (5,4)$

$c_3 + 2c_4 = 5$

$c_3 + 3c_4 = 4$

$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 1 & 3 & 4 \end{array}\right] \xrightarrow[\text{ops}]{\text{row}} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -1 \end{array}\right]$

$\boxed{\Rightarrow [v]_T = \begin{bmatrix} -7 \\ 4 \end{bmatrix} \quad \text{and} \quad [w]_T = \begin{bmatrix} 7 \\ -1 \end{bmatrix}}$

b) Find $P_{S \leftarrow T}$ (from T-basis to S-basis)

$c_1(1,2) + c_2(0,1) = (1,1)$

$c_1 = 1$

$2c_1 + c_2 = 1$

$\Rightarrow 2(1) + c_2 = 1$

$\Rightarrow c_2 = -1$

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$c_3(1,2) + c_4(0,1) = (2,3)$

$c_3 = 2$

$2c_3 + c_4 = 3$

$\Rightarrow 2(2) + c_4 = 3$

$\Rightarrow c_4 = -1$

$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\boxed{\begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = P_{S \leftarrow T}}$

c) using $P_{S \leftarrow T}$ find $[v]_S$, $[w]_S$

$[v]_S = P_{S \leftarrow T} [v]_T = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -7 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 3 \end{bmatrix} = [v]_S}$

$[w]_S = P_{S \leftarrow T} [w]_T = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ -6 \end{bmatrix} = [w]_S}$

6.7 (T1)

$S = \{v_1, \ldots, v_n\}$ basis for V ($\dim V = n$)

$v, w \in V$. Show $v = w \iff [v]_S = [w]_S$

$\mapsto$ Suppose $v = w$

Coordinate of a vector relative to a basis S are the coefficients to express vector in terms of the elements of the basis

Since a vector has a unique such expression

$\Rightarrow [v]_S = [w]_S$, as required.

$\leftarrow$ Suppose $[v]_S = [w]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

$\Rightarrow v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$

and $w = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n$

$\Rightarrow v = w$, as required $\blacksquare$