Final Exam
MAT 235B / STA 235B

1. You are allowed to use <u>ONLY</u> Durrett's textbook, Romik's notes, and your lecture notes.

2. No collaboration allowed.

3. Final Exam is due Tuesday, March 20, 2018 by 3:00 pm (return to my office or send a pdf file)

4. Exam consists of 7 problems.

#1	#2	#3	#4	#5	#6	#7	TOTAL

Problem 1

Let $G_n(s)$ be the probability generating function of the size Z_n of the n^{th} generation of a branching process (i.e. $G_n(s) = \mathbb{E}\, s^{Z_n}$), where $Z_0 = 1$ and $Var(Z_1) > 0$. Let H_n be the inverse function of G_n (we view G_n as a function on $[0,1]$).

Show that

$M_n = (H_n(s))^{Z_n}$ defines a martingale with respect to the filtration given by the sequence $\{Z_n\}_{n \geq 0}$.

Problem 2

Let $(X_n)_{n \geq 0}$ be a Markov chain. Show that, for $1 < r < n$,

$$\mathbb{P}(X_r = k \mid X_i = x_i \text{ for } i = 1, 2, \ldots, r-1, r+1, \ldots n)$$

$$= \mathbb{P}(X_r = k \mid X_{r-1} = x_{r-1}, X_{r+1} = x_{r+1})$$

Problem 3

Consider the symmetric random walk in three dimensions on $\mathbb{Z}^3$

$= \{ (x,y,z) : x,y,z = 0, \pm 1, \pm 2, \ldots \}$.

In other words, $X_0 = (0,0,0)$, and

$\mathbb{P}(X_{n+1} = X_n + \varepsilon) = \frac{1}{6}$, where

$\varepsilon = (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$.

Show that $\mathbb{P}(X_{2n} = (0,0,0)) =$

$$\left(\frac{1}{6}\right)^{2n} \sum_{i+j+k=n} \frac{(2n)!}{(i!\,j!\,k!)^2}$$

$$= \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \sum_{i+j+k=n} \left(\frac{n!}{3^n\, i!\, j!\, k!} \right)^2$$

and deduce by Stirling's formula that the origin is a transient state.

Problem 4

A function $f: S \to \mathbb{R}$ is said to be superharmonic if

$$f(x) \geq \sum_{y \in S} p(x,y) f(y), \quad \text{or}$$

equivalently, $f(X_n)$ is a supermartingale.

Suppose that p is irreducible. Show that p is recurrent iff every non-negative superharmonic function is constant.

Note: iff = if and only if.

Problem 5

Let (X_n) and (Y_n) be positive integrable and adapted to $\mathcal{F}_n$.

Suppose $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) \leq X_n + Y_n$,

with $\sum_n Y_n < \infty$ a.s.

Prove that X_n converges a.s. to a finite limit.

<u>Hint</u>: Find a supermartingale using the inequality above.

Problem 6

A particle performs a random walk on the vertex set of a connected graph G (for simplicity we assume that G does not have loops and multiple edges). At each stage it moves to a neighbor of its current position, each such neighbor is chosen with equal probability. If G has d edges, show that the stationary distribution is unique and given by

$$\pi(x) = \frac{d(x)}{2d},$$

where $d(x)$ is the degree of a vertex x.

Problem 7

Prove that if X_n and Y_n are submartingales w.r.t. $\mathcal{F}_n$ then $\max(X_n, Y_n)$ is also a submartingale.