Take Home Final Exam
MAT 235B / STA 235B

1. You are allowed to use ONLY Durrett's textbook and your lecture notes.

2. No collaboration allowed.

3. Final exams due Tuesday, March 15 by 3:00pm

4. Exam consists of six problems.

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Problem 1

Let \( \{ S_n : n \geq 0 \} \) be a simple random walk with \( S_0 = 0 \). Show that \( X_n = |S_n| \) defines a Markov chain and find the transition probabilities of this chain.

Let \( M_n = \max \{ S_k : 0 \leq k \leq n \} \). Show that \( M_n - S_n \) defines a Markov chain.

Note: By the definition of a simple random walk,

\( S_n = X_1 + \ldots + X_n \) for \( n \geq 1 \), where

\( \{ X_i \}_{i=1}^{\infty} \) are i.i.d. Bernoulli r.v.s

with \( P(X_i = 1) = p = 1 - P(X_i = -1) \).
Problem 2

Let \( X_1, X_2, \ldots \) be independent random variables such that

\[
X_n = \begin{cases} 
  a_n & \text{with prob. } \frac{1}{2n^2} \\
  0 & \text{with prob. } 1 - \frac{1}{n^2} \\
  -a_n & \text{with prob. } \frac{1}{2n^2} 
\end{cases}
\]

where \( a_1 = 2 \) and \( a_n = 4 \sum_{j=1}^{n-1} a_j \).

Show that \( Y_n = \sum_{j=1}^{n} X_j \) defines a martingale and that \( Y_\infty = \lim_{n \to \infty} Y_n \) exists almost surely.

Show that one cannot directly apply the Martingale Convergence Thm.