Problem 3

Let $X$ and $Y$ be Markov chains on the set $\mathbb{Z}$ of integers. Is the sequence $Z_n = X_n + Y_n$ necessarily a Markov chain?
Problem 4

Consider a branching process, i.e.
\[ Z_0 = 1 \] and \[ Z_{n+1} = \begin{cases} \xi_1 + \cdots + \xi_{Z_n} & \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0, \end{cases} \]

with \( \xi_i \), \( i, n \geq 0 \) being i.i.d. non-negative integer-valued random variables. Assume that \( \text{Var}(\xi_i) > 0 \). Let \( G_n(s) = \mathbb{E} s^{Z_n} \) be the probability generating function of \( Z_n \) and \( H_n \) be the inverse function of \( G_n \), viewed on \([0, 1]\), i.e.
\[ G_n(H_n(s)) = H_n(G_n(s)) = s \quad \text{for } 0 \leq s \leq 1. \]

Show that \( M_n = (H_n(s))^{Z_n} \) defines a martingale w.r.t. the natural filtration.

Hint: Recall that
\[ G_n(s) = \varphi(\varphi(\cdots \varphi(s) \cdots)), \] where
\[ \varphi(s) = \mathbb{E} s^{\xi_i} \]
Problem 5

Let $X$ be a Markov chain on a countable set $S$.

Define $T_y = \inf \{ n \geq 1 : X_n = y \}$.

Prove that

$$p^n(x, y) = \sum_{m=1}^{n} P_x(T_y = m) p^{n-m}(y, y).$$
Problem 6

Let $X$ be a discrete-time Markov chain with countable state space $S$ and transition probabilities $p(i,j), i,j \in S$. Suppose that $\Psi : S \to \mathbb{R}$ is bounded and satisfies

$$\sum_{j \in S} p(i,j) \Psi(j) \leq \lambda \Psi(i)$$

for some $\lambda > 0$ and all $i \in S$.

Show that $\lambda^{-n} \Psi(X_n)$ is a supermartingale.