Take Home Midterm Exam

MAT 235 B / STA 235 B

1. You are allowed to use ONLY Durrett's textbook and lecture notes

2. No collaboration is allowed.

3. Midterms are due Tuesday, February 13 by 10:30 a.m (in class)

4. Midterm consists of seven problems.

Problem	#1	#2	#3	#4	#5	#6	#7

Problem 1

Let $\{\xi_i\}_{i=1}^{\infty}$ be an i.i.d. sequence, N an independent positive integer-valued random variable and

$$X = \sum_{i=1}^{N} \xi_i .$$

Suppose that ξ_1 and N are integrable.

Prove that X is also integrable and compute $\mathbb{E} X$.

Problem 2

Show that if X is a non-negative supermartingale and T is a stopping time, then

$$\mathbb{E}(X_T ; T < \infty) \leq \mathbb{E}(X_0)$$

(Hint: Use Fatou's lemma)

Deduce that

$$c \operatorname{Prob}(\sup_n X_n \geq c) \leq \mathbb{E}(X_0)$$

Problem 3

(a) Show that if X is a random variable with values in $[-c,c]$ such that $\mathbb{E}(X)=0$, then for all $t \in \mathbb{R}$ one has

$$\mathbb{E}\, e^{tX} \le \cosh(tc) \le \exp\left(\tfrac{1}{2} t^2 c^2\right)$$

(Hint: Use convexity of $f(x) = \exp(tx)$ so $f(x) \le \frac{c-x}{2c} f(-c) + \frac{c+x}{2c} f(c)$).

(b) Prove that if $\{M_n\}_{n\ge 0}$ is a martingale with $M_0 \equiv 0$ and such that $|M_n - M_{n-1}| \le C_n \quad \forall n$ for some sequence of positive constants $\{C_n\}_{n\ge 1}$ then for $a > 0$

$$\mathrm{Prob}\left(\sup_{0\le k\le n} M_k \ge a\right) \le \exp\left[-\frac{1}{2} \frac{a^2}{\sum_{k=1}^{n} C_k^2}\right]$$

(Hint: Use the fact that e^{tM_n} is a submartingale and apply Doob's Inequality - see Section 5.4 in Durrett)