Problem 4

Let \( Y_1, Y_2, \ldots \) be i.i.d. random variables, \( \Pr(Y_i = 0) = \Pr(Y_i = 2) = \frac{1}{2} \)
and let \( X_n = \prod_{k=1}^{n} Y_k \).

(a) Show that \( \{X_n\}_{n \geq 1} \)
is a martingale.

(b) Show that there does not exist an integrable random variable \( X \)
and a filtration \( \{\mathcal{F}_n\}_{n \geq 1} \) such that
\[
X_n = \mathbb{E}(X | \mathcal{F}_n).
\]

This example shows that not every martingale \( \{X_n\}_{n \geq 1} \)
can be represented in the form \( \{\mathbb{E}(X | \mathcal{F}_n)\}_{n \geq 1} \).
Problem 5 (Pólya's Urn)

At time 0, an urn contains 1 black ball and 1 white ball. At each time $n = 1, 2, 3, \ldots$, a ball is chosen at random from the urn and is replaced together with a new ball of the same color. Thus, just after time $n$, there are exactly $n+2$ balls in the urn.

Let us denote by $B_n + 1$ the number of black balls in the urn just after time $n$.

Prove that for $0 < \theta < 1$,

$$N_n^\theta = \frac{(n+1)!}{B_n! (n-B_n)!} \theta^{B_n} (1-\theta)^{n-B_n}$$

is a martingale w.r.t. a natural filtration (which you should specify).
Problem 6

Suppose that $T$ is a stopping time such that for some integer $M > 0$ and some $\varepsilon > 0$, we have, for every $n$:

$$\text{Prob}(T \leq n + M | F_n) > \varepsilon \quad \text{a.s.}$$

Prove that $\mathbb{E}(T) < \infty$.

**Hint:** Prove by induction that

$$\text{Prob}(T > k M) \leq (1 - \varepsilon)^k$$

for $k = 1, 2, 3, \ldots$
Problem 7

Suppose that $X_1, X_2, \ldots$ are i.i.d. Bernoulli random variables with $\Prob(X_i = 1) = p$, $\Prob(X_i = -1) = q$, $0 < p = 1 - q < 1$, $p \neq q$.

Suppose that $a$ and $b$ are integers with $0 < a < b$. Define

$$S_n := a + X_1 + X_2 + \ldots + X_n,$$

$$T := \inf \{ m : S_m = 0 \text{ or } S_m = b \}.$$

Explain why $T$ satisfies the conditions of Problem 6 with $F_n = \sigma(X_1, X_2, \ldots, X_n)$.

Calculate $\Prob(S_T = 0)$ and $\Ex(S_T)$.

Hint: Prove that

$$M_n := \left( \frac{q}{p} \right)^{S_n} \quad (\text{and } N_n := S_n - n(p-q))$$

are martingales. The second one is useful if you are interested in $F_T$. 