Problem 4

Let $\{\xi_i\}_{i\geq 1}$ be i.i.d. random variables, $\text{Prob}(\xi_i = 0) = \text{Prob}(\xi_i = 2) = \frac{1}{2}$

and let $X_n = \prod_{k=1}^{n} \xi_k$

(a) Show that $\{X_n\}_{n\geq 1}$ is a martingale.

(B) Show that there does not exist an integrable random variable X and a filtration $\{\mathcal{F}_n\}_{n\geq 1}$ such that

$$X_n = \mathbb{E}(X \mid \mathcal{F}_n).$$

This example shows that not every martingale $\{X_n\}_{n\geq 1}$ can be represented in the form $\{\mathbb{E}(X \mid \mathcal{F}_n)\}_{n\geq 1}$.

Problem 5 (Pólya's Urn)

At time 0, an urn contains 1 black ball and 1 white ball. At each time $n = 1, 2, 3, \ldots$, a ball is chosen at random from the urn and is replaced together with a new ball of the same color. Thus, just after time n, there are exactly $n+2$ balls in the urn.

Let us denote by $B_n + 1$ the number of black balls in the urn just after time n.

Prove that for $0 < \theta < 1$,

$$N_n^\theta = \frac{(n+1)!}{B_n!\,(n-B_n)!}\,\theta^{B_n}(1-\theta)^{n-B_n}$$

is a martingale w.r.t. a natural filtration (which you should specify)

Problem 6

Suppose that T is a stopping time such that for some integer $M > 0$ and some $\varepsilon > 0$, we have, for every n:

$$\mathrm{Prob}\,(T \leq n + M \mid \mathcal{F}_n) > \varepsilon \quad \text{a.s.}$$

Prove that $\mathbb{E}(T) < \infty$.

Hint: Prove by induction that

$$\mathrm{Prob}\,(T > kM) \leq (1-\varepsilon)^k$$

for $k = 1, 2, 3, \ldots$

Problem 7

Suppose that $X_1, X_2, \ldots$ are i.i.d. Bernoulli random variables with $\text{Prob}(X_i = 1) = p$, $\text{Prob}(X_i = -1) = q$,

$$0 < p = 1 - q < 1, \quad p \neq q.$$

Suppose that a and b are integers with $0 < a < b$. Define

$$S_n := a + X_1 + X_2 + \ldots + X_n,$$

$$T := \inf\{m : S_m = 0 \text{ or } S_m = b\}.$$

Explain why T satisfies the conditions of Problem 6 with $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$.

Calculate $\text{Prob}(S_T = 0)$ and $\mathbb{E}(S_T)$.

Hint: Prove that

$$M_n := \left(\frac{q}{p}\right)^{S_n} \quad \left(\text{and } N_n := S_n - n(p-q)\right)$$

are martingales. The second one is useful if you are interested in $\mathbb{E}T$.