A crash course on "Localization".

From physics, one knows that the evolution of a single electron in the presence of a potential \( V \) is governed by the Schrödinger equation:

\[
\frac{i\hbar}{\Delta t} \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi(x,t) + eV(x)\Psi(x,t)
\]

where

- \( \hbar \) = Planck's constant.
- \( m \) = electron's mass.
- \( e \) = electron's charge.

This is typically rewritten as:

\[
\frac{\partial \Psi(x,t)}{\partial t} = \frac{\hbar^2}{2m} \Delta \Psi(x,t)
\]

acting on

\[
\hat{H} = \frac{-\hbar^2}{2m} \Delta + V
\]

where \( \hat{H} \) is the self-adjoint Hamiltonian

\[
\text{acting on } \quad H = L^2(\mathbb{R}^3, \mathbb{C})
\]

Here \( x \in \mathbb{R}^3 \)

\[
= L^2(\mathbb{R})
\]

\[
\text{for discrete}
\]

\[
-\Delta \Psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} - e^2 V\Psi
\]

Goal: Given \( V \) and an initial condition \( \Psi(x,0) \), can you solve for

\[
\Psi(t) = \Psi(x,t)
\]
Since this is a simple linear equation
the answer is yes.

\[ y_t = U(t) y_0 \]

where

\[ U(t) \]

is the unitary evolution

\[ U(t) = e^{-iHt} \]

(to be understood in terms of functional calculus)

\[ \Rightarrow \text{Time dependent problem can be solved by} \]

knowing the spectral properties of \( H \)

(i.e. a time-independent problem)

The quantum picture:

initial

Given any state of the system \( \psi_0 \in \mathcal{H} = L^2(X) \)
the probability of finding the electron in a measurable subset \( A \subset X \) at some time \( t \) is given by

\[ P(A, t) = \frac{\int_A |\psi(x, t)|^2 dx}{\int |\psi(x, t)|^2 dx} = \frac{\| \chi_A \psi_t \|^2_{L^2(X)}}{\| \psi_t \|^2_{L^2(X)}} \]

Typically one takes \( \psi_0 \in \mathcal{H} \) with \( \| \psi_0 \| = 1 \)
so that \( \| \psi_t \|^2 = \| \psi_0 \|^2 = 1 \).
The question of dynamics:

Given a self-adjoint Hamiltonian

\[ H = -\Delta + V \text{ on } L^2(\mathbb{R}) \]

and an initial condition \( \psi_0 \in L^2(\mathbb{R}) \),

How does \( \psi \) evolve under \( H \)?

**Def.** Let \( H \) be a s.a. operator on \( L^2(\mathbb{R}) \), then

1) \( \psi \in L^2(\mathbb{R}) \) is called a **bound state** for \( H \)

if \( \psi \geq 0 \) \& \( \mathcal{K} \subset \mathbb{R} \) compact s.t.

\[ \| \mathcal{K} \mathcal{K} e^{-iHt} \psi \| \leq \varepsilon \quad \forall t \in \mathbb{R} . \]

**Idea.** Suppose \( \psi_0 \in L^2(\mathbb{R}) \) is an eigenfunction of

\( H \). Then

\[ H \psi_0 = E \psi_0 \text{ for some } E \in \mathbb{R} . \]

Since \( \psi_0 \in L^2(\mathbb{R}) \):

\[ \mathbf{K} \subset \mathbb{R} : \]

\[ \| \mathbf{K} \mathcal{K} \psi_0 \| \leq \varepsilon \quad \text{and} \quad \psi_t = e^{-iHt} \psi_0 \]
and \( \| \psi_0 \| = \| \psi_1 \| \) so the initial state does not evolve under \( H \). (Only changes by a phase.)

b) \( \psi \in L^2(\mathbb{R}) \) is called a **scattering state in time mean** if for all \( K \subset \mathbb{R} \) compact, one has that

\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} \int_{K} \psi(x) e^{itH} \psi^* \, dx \, dt = 0
\]

i.e. on average, \( \psi \) evolves out of every compact set \( K \).

**Theorem:** Let \( H_b(\mathbb{R}) \) and \( H_s(\mathbb{R}) \) be the set of all bound states of \( H \) and scattering states in time mean for \( H \). Then

a) \( H_b(\mathbb{R}) \) and \( H_s(\mathbb{R}) \) are closed subspaces of \( L^2(\mathbb{R}) \).

b) \( H_b(\mathbb{R}) \perp H_s(\mathbb{R}) \)
The spectral question:

Given a self-adjoint Hamiltonian

\[ H = -\Delta + V \quad \text{on} \quad L^2(x) \]

**Def**

1) Let \( H \) be a.s.a. op. on \( L^2(x) \).
   \( \text{H} \text{pp}(H) := \) closed linear hull of eigen vectors of \( H \).

By def. \( \text{H} \text{pp}(H) \) is a closed subspace of \( L^2(x) \).

2) Set

\[ \text{H} \text{c}(H) := \text{H} \text{pp}(H) \perp \]

the orthogonal complement of \( \text{H} \text{pp}(H) \) in \( L^2(x) \).

**Theorem (RAGE)** Ruelle / Amrani / Georgescu / Eiss
Under some assumptions on \( H \) (if \( x = \mathbb{R}^3 \)) (also ass. if \( x = \mathbb{Z}^d \)) then

\[ \text{H} \text{sc}(H) = \text{H} \text{pp}(H) \]

\[ \text{H} \text{es}(H) = \text{H} \text{c}(H) \]

Relation between dynamical and spectral subspaces.

In the event that \( H = -\Delta + V \) on \( L^2(\mathbb{R}^3) \)

then the condition is that \( \gamma_k \) is an \( H \)-compact operator for every compact subset \( K \subset \mathbb{R}^3 \). i.e. \( \gamma_k(H+i)^{-1} \) is compact \( \forall K \subset \mathbb{R}^3 \) compact. (True if \( V \) is bounded.)
An aside: More on the spectral subspaces.

Suppose \( H \) is a s.a. operator on a separable Hilbert-space \( \mathcal{H} \). Let \( \{ E_x \} \) be the corresponding spectral family, i.e.,

\[
\langle H^4, 4 \rangle = \sum_x d \langle E_x^4, 4 \rangle
\]

and

\[
\langle f(H)^4, 4 \rangle = \sum_x f(x) d \langle E_x^4, 4 \rangle
\]

It can be "diagonalized" with a family of projectors.

**Definition:** For any \( 4 \in \mathcal{H} \) define the function (also measure)

\[
\mathcal{P}_4 (x) = \langle E_x^4, 4 \rangle
\]

(non-decreasing)

Hence also a measure

**Theorem:**

a) \( 4 \in \mathcal{H}_{sp}(H) \) \( \iff \) \( \exists \) countable set \( A \subset \mathbb{R} \) s.t.

\[
\text{supp}(\mathcal{P}_4) = A
\]

\[
\mathcal{P}_4[I] = \mathcal{P}_4(A)
\]

b) \( 4 \in \mathcal{H}_{c}(H) \) \( \iff \) \( A \subset \mathbb{R} \) countable

\[\Rightarrow\] \( \mathcal{P}_4(A) = 0 \).

**Definition:** Let \( H \) be s.a. operator on \( \mathcal{H} \) (separable) then the set

\[
\mathcal{H}_{sc}(H) := \{ 4 \in \mathcal{H}_{c}(H) : \exists N \subset \mathbb{R} : |N| = 0 \}
\]

and \( \text{supp}(\mathcal{P}_4) = N \).

The singular continuous subspace.
Lemma:

$H_{sc}(H)$ is a closed subspace of $H_c(H)$.

Thus we can define

$H_{ac}(H) = H_{sc}(H)^\perp \cap H_c(H)$

to be the absolutely continuous subspace corresponding to $H$.

Theorem:

$H_{ac}(H) = \{ \psi \in H : \exists \varphi \ll 1, \exists \}\}$

ie. If $N \cap R : |N| = 0$

$\Rightarrow \exists \varphi(N) = 0$.

Conclusion: Given $H$ s.a. on $H$ (separable)

$H = H_{pe} \oplus H_{ac} \oplus H_{sc}$

Hilbert space $H = H_{pe} \oplus H_{ac} \oplus H_{sc}$

Lemma: Must be invariant.

$\sigma(H) = \sigma_{pe}(H) \cup \sigma_{ac}(H) \cup \sigma_{sc}(H)$

where $\sigma_x(H) = \sigma(H \mid H_{H_x})$ These are the spectral types.
Some Interesting Hamiltonians:

- Let $\lambda \in \mathbb{R}$ be a real parameter.
- Let $p(x)$ be a probability density on $[a, b]$, i.e.

$$\int_a^b p(x)dx = \int_0^\infty p(\omega)dw = 1$$

- Let $\mathcal{N} = \prod_{n \in \mathbb{Z}^d} p(\omega_n)dw$ be the infinite product space.

In words: each $\omega_n$ is drawn according to $p(x)dx$.

Define the Hamiltonian

$$H_x(\omega) := -\Delta + \lambda \omega$$

on $L^2(\mathbb{Z}^d)$ by

$$(H_x(\omega)\psi)(\omega) = -\Delta \psi(\omega) + \lambda \omega \cdot \psi(\omega)$$

$$= -\sum_{m \in \mathbb{Z}^d: |m-n|=1} \psi(\omega) + \lambda \omega \cdot \psi(\omega)$$

This self-adjoint Hamiltonian is called a random Schrödinger operator of Anderson type. Here $\lambda \in \mathbb{R}$ is a coupling constant which determines the strength of the random potential $\lambda \omega$. 
For \( x \in \mathbb{R} \) fixed,

Note: \( \forall x \in \mathbb{R} \), \( H_x(w) \) is a s.a. Hamiltonian thus "a" random Schrödinger operator is a family of s.a. Hamiltonians parametrized by \( w \in \mathbb{R} \).

Main question: How do the spectral (and/or dynamical) properties of \( H_x(w) \) depend on \( x \)?

**Important Fact:** (Take \( d = 1 \) for simplicity)

Let \( T : \mathbb{N} \to \mathbb{N} \) be the shift
def \( (Tn)(k) = wn+1 \) (measure preserving and ergodic) and let

\( U : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \) be the shift

\[ (Uf)(n) = f(n-1) \]

Then

\[ H_x(Tn) = U H_x(w) U^{-1} \quad \forall w \in \mathbb{R} \]

i.e. this random Schrödinger operator is ergodic (i.e. \( H_x \) has a nice symmetry.)
A consequence of this symmetry (and ergodicity).

**Theorem (Pastur)**

Fix $x \in \mathbb{R}$.

There exist deterministic sets $\Xi^x$, $\Xi_{pp}$, $\Xi_{ac}$, $\Xi_{sc}$ such that:

$$\Gamma(H_x(\omega)) = \Xi^x \quad \text{for a.e. } \omega \in \mathbb{R}$$

$$T_x(H_x(\omega)) = \Xi^x \quad \text{for a.e. } \omega \in \mathbb{R}$$

for any $x \in \{pp, ac, sc\}$.

**The picture:** Suppose $\omega \in \mathcal{A} = \Pi([-1,1])$

$H_0(\omega) = -\Delta$ is non-random

$$\Gamma(H_0(\omega)) = \Gamma(-\Delta) = \Gamma_{ac}(-\Delta) = [-2d, 2d]$$

No pure point spec.

and no sing.-cont. spec.

**Graphical representation:**

$\nabla(H_x(\omega)) = \nabla_{pp}(H_x(\omega))$ a.s.

Phase transition:

spectrally:

ac $\Rightarrow$ pp.

dynamically:

transport $\Rightarrow$ no transport