As we have seen, the distribution of the eigenvalues of an \( n \times n \) random real symmetric (Hermitian) Wigner matrix

\[
A = \begin{pmatrix} a_{ij} \end{pmatrix}_{1 \leq i,j \leq n} \quad ; \quad a_{ij} = \frac{\zeta_{ij}}{\sqrt{n}},
\]

\( \mathbb{E} \zeta_{ij} = 0, \quad \text{Var} \zeta_{ij} = 1, \)

follows the Wigner semicircle law with the density

\[
f(x) = \begin{cases} \frac{1}{2 \pi} \sqrt{4 - x^2}, & |x| \leq 2, \\ 0, & |x| > 2. \end{cases}
\]

In other words,

\[
\frac{\# \{ \lambda_i(A) \in [a, b] \}}{n} \quad \xrightarrow{n \to \infty} \quad \int_{a}^{b} \frac{1}{2 \pi} \sqrt{4 - x^2} \chi_{[-2,2]}(x) \, dx \tag{40}
\]

for all \( -\infty < a < b < +\infty. \)

The next question is how rigid is the distribution of the eigenvalues.
We shall compare it with two other configurations: purely random and purely deterministic.

a) Purely Random Configuration

Let \( \{ Y_i \}_{i=1}^n \) be independent identically distributed eigenvalues with the common probability density

\[
P(x) = \frac{1}{2\pi} \sqrt{1 - x^2} X_{[0,2]}(x).
\]

Then for any \(-\infty < a < b < \infty\),

\[
\{ X_{[a,b]}(Y_i) \}_{i=1}^n
\]

are i.i.d. Bernoulli random variables s.t.

\[
\Pr \left( \bigoplus_{i=1}^n X_{[a,b]}(Y_i) = 1 \right) = \Pr \left( Y_i \in [a,b] \right) = \int_a^b p(x) \, dx,
\]

\[
\Pr \left( X_{[a,b]}(Y_i) = 0 \right) = \Pr \left( Y_i \notin [a,b] \right) = 1 - \int_a^b p(x) \, dx.
\]
Obviously, by the Law of Large Numbers,

\[
\frac{\# \{ y_i \in [a, b], 1 \leq i \leq n \}}{n} = \frac{1}{n} \sum_{i=1}^{n} \chi_{[a, b]}(y_i)
\]

\[\Rightarrow \int_{a}^{b} f(x) \, dx.\]

Moreover, by the Central Limit Theorem,

\[
\frac{\# \{ y_i \in [a, b], 1 \leq i \leq n \}}{n} = n \int_{a}^{b} f(x) \, dx
\]

\[\Rightarrow o\left(\frac{1}{\sqrt{n}}\right), \text{ and} \]

\[
\frac{\# \{ y_i \in [a, b], 1 \leq i \leq n \} - n \int_{a}^{b} f(x) \, dx}{\sqrt{n}} \Rightarrow N(0, \sigma^2)
\]
b) Purely Deterministic Case

Define 

\[-2 < x_1 < x_2 < \ldots < x_n = 2\]

to be non-random numbers such that

\[
\int_{-2}^{x_k} p(x) \, dx = n \int_{-2}^{x_k} \frac{1}{2\pi} \sqrt{4-x^2} \, dx = k,
\]

\[1 \leq k \leq n.\]

i.e.

\[
\int_{-2}^{x_k} p(x) \, dx = \frac{k}{n}.
\]

Then, it follows from the construction that for all \(-\infty < a < b < +\infty\)

\[
\left| \# \{ x_k \in [a, b], 1 \leq k \leq n \} - n \int_{a}^{b} p(x) \, dx \right| \leq \frac{2}{n}.
\]

In particular, if \(a\) and \(b\) are both in the set \(\{ x_1, x_2, \ldots, x_n \}\), then the two quantities are exactly equal.
The question now is what can we say about the remainder term in

\[ \# \{ \lambda_k(A) \in [a, b], 1 \leq k \leq n \} - n \int_a^b f(x) \, dx = o(n) \]

The remarkable fact is that the remainder term is of the order \( O\left( \sqrt{\log n} \right) \) and

\[ \# \{ \lambda_k(A) \in [a, b], 1 \leq k \leq n \} - n \int_a^b f(x) \, dx \leq \sqrt{\log n} \]

\[ \Rightarrow N(0, \sigma^2) \), where the expression for \( \sigma^2 \) is known.\]
Our goal is to explain how one can prove CLT-type result for Gaussian, Wishart, and similar ensembles that allow determinantal formulas for k-point correlation functions. We recall that if \( \{ x_i \} \) is a finite (or countable) random collection of points on the real line then its one-point correlation function \( \rho_1(x) \) is defined in such a way that

\[
E \# \{ x_i \in [a, b] \} = \int_a^b \rho_1(x) \, dx
\]

for all \(-\infty < a < b < +\infty\).
In a similar fashion, the two-point correlation function $p_2(x_1, x_2)$ is defined in such a way that

$$E \left( \text{# of pairs of particles in } [a, b] \right)$$

$$= E \left[ \# \{x_i \in [a, b]\} \cdot \left( \# \{x_i \in [a, b]\} - 1 \right) \right]$$

$$= \int_a^b \int_a^b p_2(x_1, x_2) \, dx_1 \, dx_2.$$ 

In general,

$$E \left( \text{# of } k \text{-tuples of particles in } [a, b] \right)$$

$$= E \left[ \# \{x_i \in [a, b]\} \cdot \left( \# \{x_i \in [a, b]\} - 1 \right) \cdots \left( \# \{x_i \in [a, b]\} - k + 1 \right) \right]$$

$$= \int_a^b \cdots \int_a^b p_k(x_1, \ldots, x_k) \, dx_1 \cdots dx_k,$$ where
$P_k(x_1, \ldots, x_n)$ is the $k$-point correlation function.

**Definition**

A random finite (or countable) point configuration (a.k.a. random point process) is called determinantal if there is an integral kernel $K(x, y)$ such that for any $k \geq 1$ the $k$-point correlation function $P_k(x_1, \ldots, x_n)$ has the form

$$P_k(x_1, \ldots, x_n) = \det \left( K(x_i, x_j) \right)_{1 \leq i, j \leq n}.$$