Examples

1) Gaussian Unitary Ensemble (GUE)

\[ K_n(x, y) = \sum_{l=0}^{n-1} \phi_l(x) \phi_l(y) e^{-\frac{x^2}{4}} e^{-\frac{y^2}{4}} \]

where \( \phi_l \) are orthonormal polynomials with respect to the weight \( e^{-\frac{x^2}{2}} dx \), i.e.,

\[ \int_{-\infty}^{\infty} \phi_l(x) \phi_j(x) e^{-\frac{x^2}{2}} dx = \delta_{lj} \]

\[ \text{deg} \phi_l = l, \quad l = 0, 1, 2, \ldots \]

[Can be generalized to \( P(dA) \sim e^{-\text{Tr} V(A)} dA \)]

2) Standard Wishart ensemble

\[ M = AA^* \quad A \text{ is } n \times n \text{ matrix with i.i.d. standard complex Gaussian entries} \]
Then

\[ K_n(x,y) = \sum_{l=0}^{n-1} \phi_l(x) \phi_l(y) e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}}, \]

where

\[ \int_{-\infty}^{\infty} \phi_l(x) \phi_l(x) e^{-x^2} \, dx = \delta_{l,0}, \]

i.e. \( \{ \phi_l \}_{l=0}^{\infty} \) - Laguerre polynomials.

3) Random Unitary Matrix

\( U \in U(n) \)

Haar measure

\[ e^{i \Theta_1}, e^{i \Theta_2}, \ldots, e^{i \Theta_n} \]

are the eigenvalues of a random unitary matrix \( U \) distributed uniformly according to the Haar measure.

One can view \( \Theta_1, \Theta_2, \ldots, \Theta_n \) as a random configuration on a unit circle \( S^1 = [0, 2\pi] \)
\[ K_n (\theta, \phi) = \frac{1}{2\pi} \sum_{l=0}^{n-1} \frac{e^{i l \theta}}{l} \frac{e^{i l \phi}}{l} \]

\[ = \frac{1}{2\pi} \frac{\sin \frac{n}{2} (\theta - \phi)}{\sin \frac{1}{2} (\theta - \phi)} \]

4) \[ K_{\text{sine}} (x, y) = \frac{\sin \pi (x - y)}{\pi (x - y)} \]

Useful Proposition

For a determinantal random point process,

\[ E \geq \# \text{[Eq. 63]} = \det \left( \text{Id} + (z-1) K \right) \]

where \[ K_{[a, b]} : L^2 ([a, b]) \geq (K_{[a, b]} f)(x) = \int_a^b K(x, y) f(y) dy \]
and \( \text{Det} \left( \text{Id} + (z-1) K_{[a,b]} \right) \)

\[
= \prod_{j=1}^{\infty} \left( 1 + (z-1) \lambda_j \right)
\]

is the Fredholm determinant, in other words \( \{ \lambda_j \}_{j=1}^{\infty} \) are the eigenvalues of \( K_{[a,b]} : L^2([a,b]) \to L^2([a,b]) \).

**Corollary**

\[ \# [a,b] \equiv n \text{ iff } \]

the first \( n \) eigenvalues of \( K_{[a,b]} \) are equal to 1 and the remaining eigenvalues all equal to 0.
Theorem

Suppose $X$ is a determinantal process with a trace-class kernel $K$. Write $K(x,y) = \sum_{k=1}^{n} \lambda_k \Phi_k(x) \Phi_k(y)$, where $\Phi_k$ are normalized eigenfunctions of $K$ with eigenvalues $\lambda_k \in [0,1]$. (Here $n = \infty$ is allowed).

Let $\xi_k$, $1 \leq k \leq n$ be independent random variables with $\xi_k \sim \text{Bernoulli}(\lambda_k)$.

Set $K_{\xi}(x,y) = \sum_{k=1}^{n} \xi_k \Phi_k(x) \Phi_k(y)$.

$K_{\xi}$ is a random analogue of the kernel $K$. Let $X_{\xi}$ be the determinantal process with kernel $K_{\xi}$ (i.e. first choose the $\xi_k$'s and then...
independently sample a discrete set that is determinantal with kernel $K_2$.

Then

$$X = X_2.$$

In particular, the total number of points in the process $X$ has the distribution of a sum of independent Bernoulli $(\lambda_k)$ random variables $\sum_{k=1}^n \delta_k$.

Claim

$$E \left[ \det \left( K_2 (x_i, x_i) \right)_{1 \leq i, j \leq m} \right]$$

$$= \det \left( K(x_i, x_i) \right)_{1 \leq i, j \leq m}$$
Note that \((K_{ij}(x_i, x_i))_{1 \leq i, j \leq m} = AB\),

where \(A\) is the \(m \times n\) matrix with \(A_{i,k} = \frac{1}{\sqrt{d}} \Phi_k(x_i)\) and \(B\) is the \(n \times m\) matrix with \(B_{k,j} = \Phi_k(x_i)\).

For \(A, B\) of orders \(m \times n\) and \(n \times m\), recall the Cauchy-Binet formula

\[
\det(AB) = \sum_{1 \leq i_1, i_2, \ldots, i_m \leq n} \det(A^c[i_1, \ldots, i_m]) \times \\
\det(B^c[i_1, \ldots, i_m]),
\]

where \(A^c[i_1, \ldots, i_m]\) stands for the matrix formed by taking the columns numbered \(i_1, \ldots, i_m\),
and \( B[i_1, \ldots, i_m] \) for the matrix formed by the corresponding rows of \( B \).

When we take the expectation, we have:

\[
E \left[ \det ( K_x (x_i, x_i) ) \right] = E \det (AB) = \sum_{1 \leq i_1, i_2, \ldots, i_m \leq n} E \left[ \det (A^{C_{i_1}, \ldots, i_m}) \right] \cdot \det (B^{R}(i_1, \ldots, i_m))
\]

Observe that \( B[i_1, \ldots, i_m] \) is non-random and

\[
E \left[ \det ( A^{[i_1, i_2, \ldots, i_m]} ) \right] = \det ( C^{[i_1, \ldots, i_m]} )
\]

where \( C \) is the \( m \times n \) matrix.
\[ C_{i,k} = \lambda_k \Phi_k(x_i) \]

Therefore,

\[ \mathbb{E} \left[ \det \left( K_z(x_i, x_j) \right) \right]_{1 \leq i, j \leq m} \]

\[
= \sum_{1 \leq i_1, \ldots, i_m \leq n} C^c [i_1, \ldots, i_m] B^c [i_1, \ldots, i_m] 
\]

\[
= \det (CB) = \det \left( K(x_i, x_j) \right)_{1 \leq i, j \leq m} 
\]