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GAUSSIAN LIMIT FOR DETERMINANTAL RANDOM POINT FIELDS

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We prove that, under fairly general conditions, a properly rescaled determinantal random point field converges to a generalized Gaussian random process.

1. Introduction and formulation of results. Let E be a locally compact Hausdorff space satisfying the second axiom of countability, B—σ-algebra of Borel subsets and μ a σ-finite measure on (E, B), such that $\mu(K) < \infty$ for any compact $K \subset E$. We denote by X the space of locally finite configurations of particles in E: $X = \{\xi = (x_i)_{i=-\infty}^{\infty} : x_i \in E\ \forall i$, and for any compact $K \subset E\ \#_K(\xi) := \#(x_i : x_i \in K) < +\infty\}$. A σ-algebra $\mathcal{F}$ of measurable subsets of X is generated by the cylinder sets $C_n^B = \{\xi \in X : \#_B(\xi) = n\}$, where B is a Borel set with a compact closure and $n \in \mathbb{Z}_+^1 = \{0, 1, 2, \ldots\}$. Let P be a probability measure on $(X, \mathcal{F})$. A triple $(X, \mathcal{F}, P)$ is called a random point field (process) (see [4, 17–19]). In this paper we will be interested in a special class of random point fields called determinantal random point fields. It should be noted that most, if not all the important examples of determinantal point fields arise when $E = \coprod_{i=1}^k E_i$ (here we use the notation $\coprod$ for the disjoint union), $E_i \cong \mathbb{R}^d$ or $\mathbb{Z}^d$ and μ is either the Lebegue or the counting measure. We will, however, develop our results in the general setting (our arguments will not require significant changes).

Let $dx_i,\ i = 1, \ldots, n$, be disjoint infinitesimally small subsets around the x_i's. Suppose that a probability to find a particle in each dx_i (with no restrictions outside of $\coprod_{i=1}^n dx_i$) is proportional to $\prod_{i=1}^n \mu(dx_i)$, that is,

$$P\big(\#(dx_i) = 1,\ i = 1, \ldots, n\big) = \rho_n(x_1, \ldots, x_n)\mu(dx_1)\cdots\mu(dx_n). \tag{1}$$

The function $\rho_n(x_1, \ldots, x_n)$ is then called the n-point correlation function. The equivalent definition is given by the equalities

$$E\prod_{i=1}^m \frac{(\#_{B_i})!}{(\#_{B_i} - n_i)!} = \int_{B_1^{n_1}\times\cdots\times B_m^{n_m}} \rho_n(x_1, \ldots, x_n)\, d\mu(x_1)\cdots d\mu(x_n)$$

where $B_1, \ldots, B_m$ are disjoint Borel sets with compact closures, $m \geq 1$, $n_i \geq 1,\ i = 1, \ldots, m,\ n_1 + \cdots + n_m = n$.

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A random point field is called determinantal if

$$\rho_n(x_1, \dots, x_n) = \det\bigl(K(x_i, x_j)\bigr)_{1 \le i, j \le n}, \tag{2}$$

where $K(x, y)$ is a kernel of an integral operator $K\colon L^2(E, d\mu) \to L^2(E, d\mu)$ and $K(x, y)$ satisfies some natural regularity conditions discussed below. Such a kernel $K(x, y)$ is called a correlation kernel.

It follows from (2) and the nonnegativity of the n-point correlation functions that K must have nonnegative minors, and in particular if K is Hermitian it must be a nonnegative operator. In this paper we shall always restrict ourselves to the Hermitian case.

Determinantal (also known as fermion) random point fields were introduced by Macchi in the early seventies (see [21, 22, 4]). A recent survey of the subject with applications to random matrix theory, statistical mechanics, quantum mechanics, probability theory and representation theory is given in [26]. Diaconis and Evans in [6] introduced a generalization of determinantial random point processes, called immanantal point processes.

Let K be a Hermitian, locally trace class, integral operator on $L^2(E, d\mu)$. Suppose that we can choose a kernel $K(x, y)$ in such a way that for any Borel set B with compact closure

$$\mathrm{Tr}\,(K\mathcal{X}_B) = \int_B K(x, x)\,dx, \tag{3}$$

where $\mathcal{X}_B$ denotes the multiplication operator by the indicator of B ($\equiv$projector on the subspace of the functions supported in B).

Since it is always true that

$$\begin{aligned} &\mathrm{Tr}\,(K\mathcal{X}_{B_1} \cdots K\mathcal{X}_{B_n}) \\ &\quad = \int_{B_1 \times \cdots \times B_n} K(x_1, x_2)K(x_2, x_3) \cdots K(x_n, x_1)\,d\mu(x_1) \cdots d\mu(x_n) \end{aligned} \tag{3$'$}$$

for $n > 1$ and Borel sets $B_1, \dots, B_n$ with compact closure, (3) implies that (3$'$) holds for all n.

Equation (3) can always be achieved for $E = \mathbb{R}^d$ (see, e.g., [26], Lemmas 1, 2). From now on we will assume that both (2) and (3) are satisfied.

The main goal of our paper is to study the behavior of linear statistics

$$S_f(\xi) = \sum_i f(x_i), \qquad \xi = (x_i),$$

for sufficiently "nice" test functions in a scaling limit. The moments of S_f can be calculated from (2). For instance,

$$ES_f = \int f(x)K(x, x)\,d\mu(x), \tag{4}$$

$$\mathrm{Var}\,S_f = \int f^2(x)K(x, x)\,d\mu(x) - \int f(x)f(y)|K(x, y)|^2\,d\mu(x)\,d\mu(y). \tag{5}$$

Taking $E = \mathbb{R}^1$ and $K(x, y) = \sin\pi(x - y)/\pi(x - y)$, a so-called sine kernel, we obtain a random point field well known in the theory of random matrices. It can be viewed as a limit $n \to \infty$ of the distribution of the appropriately scaled eigenvalues of $n \times n$ random Hermitian matrices with Gaussian entries (see, e.g., [5], Chapter 5). It was proven by Spohn in [29] (see also [27]), that if K is the sine kernel and a test function f is sufficiently smooth and fast decaying at infinity, then $\sum_{i=-\infty}^{\infty} f(\frac{x_i}{L}) - L\int_{-\infty}^{\infty} f(x)\,dx$ converges in distribution to the normal law $N(0, \int_{-\infty}^{\infty} |\hat{f}(k)|^2 \cdot |k|\,dk)$, where $\hat{f}$ is the Fourier transform of f, $\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,dx$. In other words we can say that the random signed measure

$$\sum_{i=-\infty}^{\infty} \delta\left(x - \frac{x_i}{L}\right) - L\,dx$$

converges as $L \to \infty$ to the generalized self-similar Gaussian random process with the spectral density $|k|$ (see, e.g., [9, 10], Section 3, [25] and, for the introduction to the theory of generalized random processes, [11]). The fact that the variance of the linear statistics $\sum_{i=-\infty}^{\infty} f(\frac{x_i}{L})$ does not grow to infinity for Schwarz functions is the manifestation of the strong repulsiveness of the distribution of the eigenvalues of random matrices. Similar results for other ensembles of random matrices have been obtained in [8, 13, 14, 1, 2, 27, 31, 7]. The kernels appearing in these ensembles are, in some respect, very much like the sine kernel. In particular, the variance of the number of particles in an interval grows as a logarithm of the mathematical expectation of the number of particles. The following result was established by Costin and Lebowitz for the sine kernel [3]: let f be an indicator of an interval, $f = \mathcal{X}_I,\ I = (a, b)$, then

$$E\sum_{i=-\infty}^{\infty} f(x_i/L) = E\big(\#(x_i : aL < x_i < bL)\big) = L(b - a),$$

$$\operatorname{Var}\left(\sum_{i=-\infty}^{\infty} f(x_i/L)\right) = \frac{1}{\pi^2}\log L + O(1)$$

and

$$\frac{\#(x_i : aL < x_i < bL) - L(b - a)}{\sqrt{(1/\pi^2)\log L}}$$

converges in distribution as $L \to \infty$ to the normal law $N(0, 1)$. The proof of the Costin–Lebowitz theorem holds, quite remarkably, for arbitrary determinantal random point fields with Hermitian kernel.

THEOREM [28]. *Let* $(X, \mathcal{F}, P_L)$, $L \geq 0$, *be a family of determinantal random point fields with Hermitian locally trace class kernels* K_L *and* $\{I_L\}_{L\geq 0}$ *be a family of Borel subsets of* E *with compact closure. Then if* $\mathrm{Var}_L(\#(x_i : x_i \in I_L)) \underset{L\to\infty}{\to} \infty$, *the normalized random variable* $(\#(x_i : x_i \in I_L) - E_L \#_{I_L})/\sqrt{\mathrm{Var}_L \#_{I_L}}$ *converges in distribution to* $N(0,1)$.

Here and below we denote by E_L, Var_L the mathematical expectation and the variance with respect to P_L. One can also establish a similar result for the step functions (finite linear combinations of indicators).

THEOREM. *Let* $(X, \mathcal{F}, P_L)$ *be a family of determinantal random point fields with Hermitian locally trace class kernels* K_L *and* $\{I_L^{(1)}, \dots, I_L^{(k)}\}_{L\geq 0}$ *be a family of Borel subsets of* E, *disjoint for any fixed* L, *with compact closure. Then if for some* $\alpha_1, \dots, \alpha_k \in \mathbb{R}^1$, *the variance of the linear statistics* $\sum_{i=-\infty}^{\infty} f_L(x_i)$ *with* $f_L(x) = \sum_{j=1}^{k} \alpha_j \cdot \mathcal{X}_{I_L^{(j)}}(x)$, *grows to infinity in such a way that* $\mathrm{Var}_L(\#_{I_L^{(j)}}) = O(\mathrm{Var}_L(\sum_{i=-\infty}^{\infty} f_L(x_i)))$ *for any* $1 \leq j \leq k$, *the central limit theorem holds*:

$$\frac{\sum_{j=1}^{k} \alpha_j^{(L)} \cdot \#_{I_L^{(j)}} - E_L\bigl(\sum_{j=1}^{k} \alpha_j \cdot \#_{I_L^{(j)}}\bigr)}{\sqrt{\mathrm{Var}_L\bigl(\sum_{j=1}^{k} \alpha_j \cdot \#_{I_L^{(j)}}\bigr)}} \xrightarrow{w} N(0,1).$$

REMARK 1. We use standard notation $f = O(g)$ and $f = o(g)$ when f/g stays bounded or $f/g \to 0$.

REMARK 2. The last theorem has been explicitly stated in [28] only in the special case of the Airy and Bessel kernels and the kernels arising in the classical compact groups (see Theorems 1, 2, 4, 6); however, the key Lemmas 7 and 8 proven there allow rather straightforward generalization to the case of an arbitrary Hermitian kernel. A result close to our Theorem 6 from [28] was also established by K. Wieand [31].

We recall that a Hermitian kernel $K(x, y)$ defines a determinantal random point field if and only if the integral operator K is nonnegative and bounded from above by the identity

$$0 \leq K \leq \mathrm{Id} \tag{6}$$

([26], Theorem 3). For the translation-invariant kernels $K(x - y)$ and $E = \mathbb{R}^d$ or $\mathbb{Z}^d$ this is equivalent to $0 \leq \hat{K}(t) \leq 1$, where

$$K(x) = \int e^{2\pi i (x\cdot t)} \hat{K}(t)\, dt. \tag{7}$$

The sine kernel $K(x - y) = \sin \pi(x - y)/\pi(x - y)$ corresponds to

$$\hat{K}(t) = \mathcal{X}_{[-1/2,1/2]}(t),$$

the indicator of $[-1/2, 1/2]$. It might be worth noting and actually is not very difficult to see, that the logarithmic rate of the growth of $\mathrm{Var}(\#(x_i : |x_i| \le L))$ for the sine kernel is the slowest among all translation-invariant kernels corresponding to projectors, $\hat{K} = \mathcal{X}_B$, for which $\inf(B)$ and $\sup(B)$ are the density points of B. For the generic translation-invariant kernel $K(x-y)$ ($\hat{K}$ is not an indicator) $\mathrm{Var}(\#(x_i : |x_i| \le L))$ is proportional to $\mathrm{Vol}(x_i : |x_i| \le L) \sim E(\#(x_i : |x_i| \le L))$ ([26], Section 3).

In our main result we prove CLT for the linear statistics when the variance grows faster than some arbitrary small, but fixed, power of the mathematical expectation.

THEOREM 1. *Let* $(X, \mathcal{F}, P_L)$, $L \ge 0$, *be a family of determinantal random point fields with Hermitian correlation kernels* K_L. *Suppose that* $f_L, L \ge 0$, *are bounded measurable functions with precompact support* [*i.e.*, $\sup(f_L)$ *has a compact closure for any* $L \ge 0$], *such that*

$$\mathrm{Var}_L\, S_{f_L} \to \infty \qquad \textit{as } L \to \infty \tag{8}$$

and

$$\sup|f_L(x)| = o(\mathrm{Var}_L)^{\varepsilon}, \qquad E_L S_{|f|_L} = O\big((\mathrm{Var}_L\, S_{f_L})^{\delta}\big), \tag{9}$$

for any $\varepsilon > 0$ *and some* $\delta > 0$. *Then the normalized linear statistics* $\frac{S_{f_L} - E_L S_{f_L}}{\sqrt{\mathrm{Var}_L\, S_{f_L}}}$ *converges in distribution to the standard normal law* $N(0,1)$.

As a very important special case of Theorem 1 one can consider $f_L(x) := f(T_L x)$, where $\{T_L\}$, $L \in \mathbb{R}^1_+$, is a one-parameter family of measurable transformations $T_L\colon E \to E$ such that $T_L^{-1} D$ has compact closure for any compact D. If for a sufficiently rich class of test functions f (e.g., continuous functions with compact support) (8), (9) are satisfied, and the rate of the growth of $\mathrm{Var}_L(S_{f_L})$ is the same,

$$\mathrm{Var}_L(S_{f_L}) = B(f) \cdot V_L \cdot \big(1 + o(1)\big),$$

where $B(f)$ is some functional on a space of test functions, Theorem 1 implies that the random signed measure

$$V_L^{-1/2}\left(\sum_{i=-\infty}^{\infty} \delta(x - T_L x_i) - T_L\big(K_L(x,x)\, d\mu(x)\big)\right)$$

converges as $L \to \infty$ to the generalized Gaussian process with the correlation functional $B(f,f) = B(f)$ [we denote by $T_L(K_L(x,x)\, d\mu(x))$ the image of the measure $K_L(x,x)\, d\mu(x)$ under T_L].

Let us consider a Euclidean one-particle space $E = \mathbb{R}^d$, a one-parameter family of dilations

$$T_L\colon \mathbb{R}^d \to \mathbb{R}^d, \qquad T_L x = x/L,$$

and a correlation kernel

$$K_L(x, y) = A_L(x - y) + R_L(x, y), \tag{10}$$

where

$$|R_L(x, y)| \le Q(x_{abs} + y_{abs}), \tag{11}$$

$x_{abs} = (|x_1|, \dots, |x_d|)$, $Q \in L^2(\mathbb{R}^d_+) \cap L^\infty(\mathbb{R}^d_+)$. It follows from (6), (10) and (11) that $0 \le A_L \le \mathrm{Id}$, which implies $0 \le \hat{A}_L(k) \le 1$, $0 \le \int_{\mathbb{R}^d} \hat{A}_L(k) - (\hat{A}_L(k))^2\, dk = A_L(0) - \int_{\mathbb{R}^d} |A_L(x)|^2\, dx =: \sigma_L^2$, and $\sigma_L = 0$ if and only if $\hat{A}_L$ is an indicator.

THEOREM 2. *Let the kernel K_L satisfy* (10), (11) *and there exist constants* const, $\sigma > 0$ *and $\kappa_L \to \infty$ as $L \to \infty$ such that*

$$\sigma_L \to \sigma \qquad \text{as } L \to \infty, \qquad |A_L(0)| < \text{const}$$

and

$$\int_{|x| > L/\kappa_L} |A_L(x)|^2\, dx \to 0.$$

Then for any real-valued function $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ the normalized linear statistics

$$\frac{1}{L^{d/2}\sigma}\left(\sum_{i=-\infty}^{\infty} f\left(\frac{x_i}{L}\right) - A_L(0) \cdot L \int_{\mathbb{R}^d} f(x)\, dx\right)$$

converges in distribution to the Gaussian random variable $N(0, \int_{\mathbb{R}^d} (f(x))^2\, dx)$.

REMARK 3. Theorem 2 says that under the stated conditions the random signed measure

$$\frac{1}{L^{d/2}\sigma}\left(\sum_{i=-\infty}^{\infty} \delta\left(x - \frac{x_i}{L}\right) - A_L(0) \cdot L\, dx\right)$$

converges to the white noise as $L \to \infty$ (for the definition of the white noise; see, e.g., [12]). Similar results hold in the discrete case.

Let us now restrict our attention to the translation-invariant kernels $K(x, y) = A(x - y)$. We will use the notation

$$m(\lambda) := \int \hat{A}(k) - \hat{A}(k)\hat{A}(k - \lambda)\, dk.$$

Observe that $\sigma^2 = m(0)$ and

$$\mathrm{Var}\left(\sum_{i=-\infty}^{\infty} f(x_i)\right) = \int |\hat{f}(\lambda)|^2 m(\lambda)\, d\lambda. \tag{12}$$

In particular

$$\operatorname{Var}\bigl(\#_{[-L,L]^d}\bigr) = \operatorname{Vol}\bigl([-L,L]^d\bigr)\cdot\bigl(m(0)+o(1)\bigr). \tag{13}$$

It follows from (12) that the rate of the growth of the variance of S_{f_L} depends on the asymptotics of $m(\lambda)$ near the origin. In the next theorem we consider the degenerate case $\sigma^2 = 0$ in one dimension.

THEOREM 3. *Let $K(x,y) = A(x-y)$ be a translation-invariant kernel in $\mathbb{R}^1$ and $m(\lambda) = |\lambda|^\alpha \varphi(\lambda)$, where $\varphi(\lambda)$ is a slowly varying function at the origin and $0 < \alpha < 1$. Then for any Schwarz function $f : ES_{f_L} = LA(0)\int f(x)\,dx$, $\operatorname{Var} S_{f_L} = L^{1-\alpha}\varphi(L^{-1})\int |\hat f(k)|^2|k|^\alpha\,dk(1+o(1))$, and*

$$\frac{S_{f_L} - ES_{f_L}}{(L^{1-\alpha}\varphi(L^{-1}))^{1/2}}$$

converges in distribution to $N(0, \int |\hat f(k)|^2|k|^\alpha\,dk)$.

REMARK 5. We recall that $\varphi(\lambda) \ge 0$ is slowly varying at the origin if $\lim_{\lambda\to 0}\frac{\varphi(a\lambda)}{\varphi(\lambda)} = 1$ for any $a \ne 0$ (see [24]).

REMARK 6. The result of Theorem 3 can be interpreted as the convergence in distribution of the random signed measure

$$\bigl(L^{1-\alpha}\varphi(L^{-1})\bigr)^{-1/2}\left(\sum \delta\left(x - \frac{x_i}{L}\right) - A(0)L\,dx\right)$$

to the self-similar (also called automodel in the Russian literature) generalized Gaussian random process with the spectral density $|k|^\alpha,\ 0<\alpha<1$. Self-similarity means that the distribution of the process is invariant under the action of the renorm-group $\xi(x) \to \xi(ax)a^\gamma$, $\gamma = (1+\alpha)/2$. The self-similar generalized Gaussian random process corresponding to $\alpha = 0$ is exactly the white noise (see Remark 3 above). It was proven by Dobrushin that the only self-similar random processes in $\mathbb{R}^1$ are the ones with the spectral density $|k|^\alpha,\ 0 \le \alpha \le 1$. A self-similar generalized random process with the spectral density $|k|$ appeared in the Spohn's results [29] discussed above after the formulas (4), (5) (see also [13, 1, 27]). For additional information on self-similar random processes we refer the reader to [9, 10, 25].

EXAMPLE. Let $\hat A$ be the indicator of $\coprod_{n\ge 1}[n, n+n^{-\beta}]$, $\beta > 1$. Then $m(\lambda) = \text{const}\cdot|\lambda|^{1-1/\beta}(1+o(1))$. On the other hand, if the length l_n of the nth interval $[n, n+l_n]$ decays sufficiently fast, say $0 \le l_{n+1} \le l_n^{1+\varepsilon}, \varepsilon > 0$, than $m(\lambda)$ is not regularly varying at the origin.

Finally we consider the case when $\hat A$ is the indicator of a union of $1 \le \ell < \infty$ disjoint intervals. It is straightforward to see that then $m(\lambda) = \ell|\lambda|$ near the origin.

THEOREM 4. *Let $\hat{A}$ be the indicator of I, $I = \coprod_{i=1}^{\ell}[a_i, b_i]$, $a_1 < b_1 < a_2 < b_2 < \cdots < a_\ell < b_\ell$. Then for any Schwarz function f, $\sum_{i=-\infty}^{\infty} f\left(\frac{x_i}{L}\right) - A(0)\cdot L \int_{-\infty}^{\infty} f(x)\,dx$ converges in distribution to $N(0, \ell \cdot \int_{-\infty}^{\infty} |\hat{f}(k)|^2 |k|\,dk)$.*

The proofs of Theorems 1–3 will be given in the next three sections. The proof of Theorem 4 is the same, modulo trivial alterations, as the one given for the sine kernel in [27].

2. Proof of Theorem 1. We are going to prove Theorem 1 by the method of moments. Let us denote by $C_n(S_f)$ the nth cumulant of S_f. We remind the reader that for a random variable η with all finite moments, the cumulants $C_n(\eta)$, $n = 1, 2, \ldots$, are defined through the Taylor coefficients of the logarithm of the characteristic function:

$$\log E\big(\exp(it\eta)\big) = \sum_{n=1}^{\infty} C_n(\eta)(it)^n/n!.$$

We show that the nth cumulant of the normalized linear statistics $(S_{f_L} - ES_{f_L})/(\mathrm{Var}\, S_{f_L})^{1/2}$ converges to zero as $L \to \infty$ for sufficiently large n ($n > \max(2\delta, 2)$). The Lemma 3 from the Appendix then asserts that all cumulants of $(S_{f_L} - ES_{f_L})/(\mathrm{Var}\, S_{f_L})^{1/2}$ converge to the cumulants of the standard normal distribution, which implies the weak convergence.

We recall the lemma established in [27] [see formula (2.7)].

LEMMA 1.

$$\begin{aligned} C_n(S_f) = \sum_{m=1}^{n} \sum_{\substack{(n_1,\ldots,n_m): n_1+\cdots+n_m=n,\\ n_i\geq 1,\ i=1,\ldots,m}} \frac{(-1)^{m-1}}{m} \frac{n!}{n_1!\cdots n_m!} \\ \times \int f^{n_1}(x_1)K(x_1,x_2)f^{n_2}(x_2)K(x_2,x_3)\cdots f^{n_m}(x_m) \\ \times K(x_m,x_1)\,d\mu(x_1)\cdots d\mu(x_m). \end{aligned} \tag{14}$$

Using Lemma 1 we will be able to estimate the cumulants of S_{f_L}. We claim the following result to be true.

LEMMA 2. *Under the assumptions of Theorem* 1,

$$C_n(S_{f_L}) = O\big((\mathrm{Var}_L\, S_{f_L})^{\delta+\varepsilon}\big), \qquad n \geq 1, \tag{15}$$

where ε is arbitrarily small.

PROOF. It follows from (14) that $C_n(S_{f_L})$ is a linear combination of

$$\begin{aligned}\int f_L^{n_1}(x_1)K_L(x_1,x_2)f_L^{n_2}(x_2)K_L(x_2,x_3)\cdots\\ \times f_L^{n_m}(x_m)K_L(x_m,x_1)\,d\mu(x_1)\cdots d\mu(x_m)\\ =\mathrm{Tr}\,(f_L^{n_1}K_Lf_L^{n_2}K_L\cdots f_L^{n_m}K_L),\end{aligned}$$

where $n_i\ge 1,\ i=1,\dots,m,\ m\ge 1$.

We claim that each term is $O((\mathrm{Var}_L\,S_{f_L})^{\delta+\varepsilon})$. Indeed, if $m=1$, then $|\,\mathrm{Tr}\ f_L^nK_L|=|\int f_L^n(x)K_L(x,x)\,d\mu(x)|\le\|f_L\|_\infty^{n-1}\int|f_L(x)|K_L(x,x)\,d\mu(x)=\|f_L\|_\infty^{n-1}ES_{|f|_L}=O((\mathrm{Var}_L\,S_{f_L})^{\delta+\varepsilon})$.

If $m>1$, represent $\mathrm{Tr}\,(f_L^{n_1}K_Lf_L^{n_2}K_L\cdots f_L^{n_m}K_L)$ as a linear combination of $\mathrm{Tr}\,(f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_2}K_L\cdots f_{\pm,L}^{n_m}K_L)$, where we use the notations $f_+=\max(f,0)$, $f_-=\max(-f,0)$. Let us fix the choice of $\pm$ in each of the factors. Using the cyclicity of the trace and the inequality $|\,\mathrm{Tr}\,(AB)|\le(\mathrm{Tr}\,(AA^*))^{1/2}(\mathrm{Tr}\,(BB^*))^{1/2}$ for the Hilbert–Schmidt operators ([RS], section VI.6), we obtain

$$\begin{aligned}&\bigl|\mathrm{Tr}\,\bigl(f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_2}K_L\cdots f_{\pm,L}^{n_m}K_L\bigr)\bigr|\\ &\quad=\bigl|\mathrm{Tr}\,\bigl(f_{\pm,L}^{n_1/2}K_Lf_{\pm,L}^{n_2}K_L\cdots f_{\pm,L}^{n_m}K_Lf_{\pm,L}^{n_1/2}\bigr)\bigr|\\ &\quad\le\bigl[\mathrm{Tr}\,\bigl(\bigl(f_{\pm,L}^{n_1/2}K_Lf_{\pm,L}^{n_2/2}\bigr)\bigl(f_{\pm,L}^{n_1/2}K_Lf_{\pm,L}^{n_2/2}\bigr)^*\bigr)\bigr]^{1/2}\\ &\qquad\times\bigl[\mathrm{Tr}\,\bigl(\bigl(f_{\pm,L}^{n_2/2}K_Lf_{\pm,L}^{n_3}K_L\cdots f_{\pm,L}^{n_m}K_Lf_{\pm,L}^{n_1/2}\bigr)\\ &\qquad\quad\times\bigl(f_{\pm,L}^{n_2/2}K_Lf_{\pm,L}^{n_3}K_L\cdots f_{\pm,L}^{n_m}K_Lf_{\pm,L}^{n_1/2}\bigr)^*\bigr)\bigr]^{1/2}.\end{aligned}\tag{16}$$

The first factor at the r.h.s. of (16) is equal (again by the cyclicity of the trace) to $[\mathrm{Tr}\,(f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_2}K_L)]^{1/2}$ [in particular we note that $\mathrm{Tr}\,(g_1Kg_2K)\ge 0$ for non-negative $g_1,\ g_2$].

Since

$$\begin{aligned}\mathrm{Tr}\,\bigl(\bigl(f_{\pm,L}^{n_1}+f_{\pm,L}^{n_2}\bigr)^2K\bigr)-\mathrm{Tr}\,\bigl(\bigl(f_{\pm,L}^{n_1}+f_{\pm,L}^{n_2}\bigr)K\bigl(f_{\pm,L}^{n_1}+f_{\pm,L}^{n_2}\bigr)K\bigr)\\ =\mathrm{Var}\bigl(S_{f_{\pm,L}^{n_1}+f_{\pm,L}^{n_2}}\bigr)\ge 0,\end{aligned}$$

we have

$$\begin{aligned}0&\le\mathrm{Tr}\,\bigl(f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_2}K_L\bigr)\\ &\le\frac12\Bigl(\mathrm{Tr}\,\bigl(\bigl(f_{\pm,L}^{n_1}+f_{\pm,L}^{n_2}\bigr)^2K_L\bigr)\\ &\qquad-\mathrm{Tr}\,\bigl(f_{\pm,L}^{n_1}K_Lf_{\pm,L}^{n_1}K_L\bigr)-\mathrm{Tr}\,\bigl(f_{\pm,L}^{n_2}K_Lf_{\pm,L}^{n_2}K_L\bigr)\Bigr)\\ &\le\frac12\,\mathrm{Tr}\,\bigl(\bigl(f_{\pm,L}^{n_1}+f_{\pm,L}^{n_2}\bigr)^2K_L\bigr)\bigr)\\ &=O\bigl(\mathrm{Tr}\,\bigl(|f_L|K_L\bigr)\bigr)o\bigl((\mathrm{Var}_L\,S_{f_L})^{\varepsilon}\bigr)\\ &=O\bigl((\mathrm{Var}_L\,S_{f_L})^{\delta+\varepsilon}\bigr).\end{aligned}\tag{17}$$

As for the second term in (16), one can rewrite $\operatorname{Tr}((f_{\pm,L}^{n_2/2} K_L f_{\pm,L}^{n_3} K_L \cdots f_{\pm,L}^{n_m} K_L f_{\pm,L}^{n_1/2})(f_{\pm,L}^{n_2/2} K_L f_{\pm,L}^{n_3} K_L \cdots f_{\pm,L}^{n_m} K_L f_{\pm,L}^{n_1/2})^*)$ as

$$\begin{aligned}(18)\qquad &\operatorname{Tr}\bigl(f_{\pm,L}^{\frac{n_2}{2}} K_L f_{\pm,L}^{n_3} K_L \cdots f_{\pm,L}^{n_m} K_L f_{\pm,L}^{n_1} K_L f_{\pm,L}^{n_m} \cdots K_L f_{\pm,L}^{n_3} K_L f_{\pm,L}^{\frac{n_2}{2}}\bigr)\\ &\quad = \operatorname{Tr}(CDD^*),\end{aligned}$$

where $C = f_{\pm}^{n_3/2} K_L f_{\pm,L}^{n_2} K_L f_{\pm,L}^{n_3/2}$, $D = f_{\pm,L}^{n_3/2} K_L f_{\pm,L}^{n_4} K_L \cdots f_{\pm,L}^{n_m} K_L f_{\pm,L}^{n_1/2}$. Note that $C \geq 0$ and $\operatorname{Tr}(C) = \operatorname{Tr}(f_{\pm,L}^{n_3} K_L f_{\pm,L}^{n_2} K) = O((\operatorname{Var}_L S_{f_L})^{\delta+\varepsilon})$ by arguments similar to (17). Using $|\operatorname{Tr}(CDD^*)| \leq \operatorname{Tr}(C) \cdot \|DD^*\| = \operatorname{Tr}(C) \cdot \|D\|^2$ ([23], Section VI.6) and $\|D\| \leq \|K\|^m \cdot \|f_L\|_\infty^{\aleph}$, where $\aleph = (\sum_{i=1}^m n_1) - n_2$, we conclude that (18) is $O((\operatorname{Var}_L S_{f_L})^{\delta+\varepsilon})$. Together with (16) and (17) this concludes the proof of the lemma. □

Let us now apply Lemma 2 to estimate the cumulants of the normalized linear statistics. We have

$$C_1\biggl(\frac{S_{f_L} - ES_{f_L}}{\sqrt{\operatorname{Var}_L S_{f_L}}}\biggr) = 0, \qquad C_2\biggl(\frac{S_{f_L} - ES_{f_L}}{\sqrt{\operatorname{Var}_L S_{f_L}}}\biggr) = 1$$

and, for $n > 2$,

$$C_n\biggl(\frac{S_{f_L} - ES_{f_L}}{\sqrt{\operatorname{Var}_L S_{f_L}}}\biggr) = \frac{C_n(S_{f_L})}{(\operatorname{Var}_L S_{f_L})^{n/2}} = O\biggl(\frac{E(S_{|f|_L})}{(\operatorname{Var}_L S_{f_L})^{n/2}}\biggr). \tag{19}$$

It follows from the Lemma 2 and (19) that

$$C_n\biggl(\frac{S_{f_L} - ES_{f_L}}{\sqrt{\operatorname{Var} S_{f_L}}}\biggr)$$

goes to zero if $n > 2\delta$.

Lemma 3 from Appendix then implies that all cumulants of the normalized linear statistics converge to the cumulants of the standard normal random variable, and weak convergence of the distributions follows.

Theorem 1 is proven. □

3. Proof of Theorem 2. Let $(E, d\mu)$ be $(\mathbb{R}^d, dx)$ and $T_L x = x/L$. Consider a real-valued function $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. The mathematical expectations of S_{f_L} is equal to

$$\begin{aligned} ES_{f_L} &= \int_{\mathbb{R}^d} f(x/L) K_L(x,x)\,dx\\ &= \int_{\mathbb{R}^d} f(x/L) A_L(0)\,dx + \int_{\mathbb{R}^d} f(x/L) R_L(x,x)\,dx\\ &= A_L(0) L^d \int f(x)\,dx + \int f(x/L) R_L(x,x)\,dx.\end{aligned}$$

By (11), the absolute value of the second integral is bounded by the sum of the integrals

$$\int_{\mathbb{R}_+^d} \big|f(\pm x_1/L,\dots,\pm x_d/L)\big| Q(2x_1,\dots,2x_d)\,dx$$
$$\le \left(\int_{\mathbb{R}_+^d} f^2(\pm x_1/L,\dots,\pm x_d/L)\,dx\right)^{1/2}\left(\int_{\mathbb{R}_+^d} Q^2(2x)\,dx\right)^{1/2} = O\big(L^{d/2}\big).$$

Therefore,

$$ES_{f_L} = A_L(0)L^d \int_{\mathbb{R}^d} f(x)\,dx + O\big(L^{d/2}\big). \tag{20}$$

The variance of S_{f_L} is given by

$$\begin{aligned}\operatorname{Var} S_{f_L} &= \int f^2(x/L)K_L(x,x)\,dx - \int f(x/L)f(y/L)|K_L(x,y)|^2\,dx\,dy\\ &= A_L(0)L^d \int f^2(x)\,dx - \int f(x/L)f(y/L)|A_L(x-y)|^2\,dx\,dy + r(L),\end{aligned} \tag{21}$$

where

$$\begin{aligned}r(L) &= \int f^2(x/L)R_L(x,x)\,dx - 2\int f(x/L)f(y/L)A_L(x-y)R_L(y,x)\,dx\,dy\\ &\quad - \int f(x/L)f(y/L)|R_L(x,y)|^2\,dx\,dy\\ &= r_1(L)+r_2(L)+r_3(L).\end{aligned}$$

It follows from the assumptions of the theorem that the second term at the r.h.s. of (21) is equal to

$$\begin{aligned}L^d \int |\hat f(k)|^2 \widehat{|A_L|^2}(k/L)\,dk &= L^d \widehat{|A_L|^2}(0) \int |\hat f(k)|^2\,dk\big(1+o(1)\big)\\ &= L^d \int |A_L(x)|^2\,dx \int f^2(x)\,dx\big(1+o(1)\big).\end{aligned}$$

Indeed,

$$\left|\int |\hat f(k)|^2\big(\widehat{|A_L|^2}(k/L) - \widehat{|A_L|^2}(0)\big)\,dk\right| \le \left|\int_{|k|>\kappa_L}\right| + \left|\int_{|k|\le\kappa_L}\right|.$$

Since

$$\big|\widehat{|A_L|^2}(t)\big| = \left|\int \hat A_L(k)\hat A_L(k-t)\,dk\right| \le \int \hat A_L(k)\,dk = A_L(0) \le \text{const}$$

we note that the first integral is bounded from above by

$$\text{const}\int_{|k|>(\kappa_L)^{1/2}} |\hat{f}(k)|^2\,dk \to 0 \qquad \text{as } L\to\infty.$$

To deal with the second integral we estimate from above

$$\begin{aligned}
&\bigl|\widehat{|A_L|^2}(k/L)-\widehat{|A_L|^2}(0)\bigr|\\
&\quad\le \left|\int |A_L|^2(t)\bigl(\exp(2\pi i t k/L)-1\bigr)\,dt\right|\left|\int_{|t|\ge L/\kappa_L}+\int_{|t|<L/\kappa_L}\right|\\
&\quad\le \int_{|t|\ge L/\kappa_L} |A_L|^2(t)\,dt + O(1/\sqrt{\kappa_L})\\
&\quad= o(1)+O(1/\sqrt{\kappa_L}) = o(1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{Var} S_{f_L} &= \left(A_L(0)-\int |A_L(x)|^2\,dx\right)L^d\int f^2(x)\,dx + o(L^d)+r(L)\\
&= \sigma^2 L^d\int f^2(x)\,dx + o(L^d)+r(L).
\end{aligned}\tag{22}$$

We claim that

$$r(L)=o(L^d).\tag{23}$$

Consider first $r_1(L)$. By (11) it is bounded by the integrals

$$\int_{\mathbb{R}^d_+} f^2(\pm x_1/L,\dots,\pm x_d/L)Q(2x)\,dx.$$

All of these integrals are estimated in the same way. For example,

$$\begin{aligned}
\int_{\mathbb{R}^d_+} f^2(x/L)Q(2x)\,dx &= L^d\int f^2(x)Q(2Lx)\,dx\\
&= L^d\int f^2(x)Q(2Lx)\mathcal{X}_{\{Q(2Lx)>1/\sqrt{L}\}}\,dx\\
&\quad + L^d\int f^2(x)Q(2Lx)\mathcal{X}_{\{Q(2Lx)\le 1/\sqrt{L}\}}\,dx\\
&\le L^d\|Q\|_\infty\int f^2(x)\mathcal{X}_{\{Q(2Lx)>1/\sqrt{L}\}}\,dx\\
&\quad + L^{d-1/2}\int f^2(x)\,dx = o(L^d),
\end{aligned}$$

since

$$\ell\big(x : Q(2Lx) > 1/\sqrt{L}\big) \underset{L\to\infty}{\to} 0.$$

To estimate $r_3(L)$ we need to estimate the integrals of the form

$$\int_{\mathbb{R}^d_+} |f(x/L)|\,|f(y/L)|Q^2(x+y)\,dx\,dy = L^d \int_{\mathbb{R}^d_+} g(z/L)Q^2(z)\,dz, \tag{24}$$

where $g(z) = \int |f(x)|\,|f(z-x)|\mathcal{X}_{\mathbb{R}^d_+}(x)\mathcal{X}_{\mathbb{R}^d_+}(z-x)\,dx$.

Since $g(z)$ is bounded, continuous, and zero at the origin, we have

$$(24) = L^d g(0) \int Q^2(z)\,dz\big(1+o(1)\big) = o(L^d).$$

Finally,

$$\begin{aligned}
|r_2(L)| &= \left| \int f(x/L) f(y/L) A_L(x-y) R(y,x)\,dx\,dy \right| \\
&\le \left[\int |f(x/L)|\,|f(y/L)|\,|A_L(x-y)|^2\,dx\,dy \right]^{1/2} \\
&\qquad \times \left[\int |f(x/L)|\,|f(y/L)|\,|R_L(y,x)|^2\,dx\,dy \right]^{1/2} \\
&= O(L^{d/2})\,o(L^{d/2}) = o(L^d).
\end{aligned}$$

Combining the above estimates, we prove (23), which implies

$$\operatorname{Var} S_{f_L} = \sigma^2 L^d \int f^2(x)\,dx\,\big(1+o(1)\big). \tag{25}$$

If f is bounded, the central limit theorem then follows from Theorem 1 (compactness of the support of f is not needed since $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ guarantees that all moments of S_{f_L} are finite). The proof in the case of the unbounded f follows by a rather standard approximation argument. We choose $N > 0$ to be sufficiently large and consider a truncated function

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } |f(x)| \le N, \\ N, & \text{if } f(x) > N, \\ -N, & \text{if } f(x) < -N. \end{cases}$$

Observe that

$$E\left(\frac{S_{f_L} - ES_{f_L}}{\sigma L^{d/2}} - \frac{S_{\tilde{f}_L} - ES_{\tilde{f}_L}}{\sigma L^{d/2}} \right)^2 = \frac{\operatorname{Var} S_{(f-\tilde{f})_L}}{\sigma^2 L^d} = \frac{\int_{|x|\ge N} f^2(x)\,dx}{\sigma^2} + o(1)$$

can be made arbitrarily small by choosing N and L sufficiently large.

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Since

$$\frac{S_{\tilde f_L} - ES_{\tilde f_L}}{\sigma L^{d/2}} \xrightarrow[L\to\infty]{w} N\left(0, \int_{|x|\le N} f^2(x)\,dx\right)$$

and

$$\lim_{N\to\infty} \int_{|x|\le N} f^2(x)\,dx = \int f^2(x)\,dx,$$

the result follows. Theorem 2 is proven. $\square$

4. Proof of Theorem 3. We now turn to the proof of Theorem 3. It is enough to establish that

$$\operatorname{Var} S_{f_L} = L^{1-\alpha}\varphi(L^{-1}) \int |\hat f(k)|^2\, |k|^\alpha\, dk\big(1+o(1)\big). \tag{26}$$

The result then will follow from Theorem 1. We have [see (12)]

$$\begin{aligned} \operatorname{Var} S_{f_L} &= \int |\hat f(L\lambda)|^2 L^2 m(\lambda)\,d\lambda \\ &= L\int |\hat f(k)|^2 m(kL^{-1})\,dk \\ &= L\int |\hat f(k)|^2\,|k|^\alpha L^{-\alpha}\varphi(kL^{-1})\,dk \\ &= L^{1-\alpha}\varphi(L^{-1})\int |\hat f(k)|^2\, |k|^\alpha \frac{\varphi(kL^{-1})}{\varphi(L^{-1})}\,dk. \end{aligned} \tag{27}$$

It was proven by Karamata ([15, 16]) that any slowly varying function at the origin can be represented in some interval $(0, b]$ as

$$\varphi(x) = \exp\left\{\eta(x) + \int_{b^{-1}}^{x^{-1}} \frac{\varepsilon(t)}{t}\,dt\right\}, \tag{28}$$

where η is a bounded measurable function on $(0, b]$, such that $\eta(x) \to c$ as $x \to 0$ ($|c| < \infty$), and $\varepsilon(x)$ is a continuous function on $(0, b]$ such that $\varepsilon(x) \to 0$ as $x \to 0$ (for a modern day reference we refer the reader to [24], Theorem 1.2; of course a similar representation holds for φ also on some interval $[b', 0)$ of the negative semiaxis). In particular

$$\frac{\varphi(k/L)}{\varphi(1/L)} \underset{L\to\infty}{\to} 1 \tag{29}$$

uniformly in k on compact subsets of $\mathbb{R}^1 \setminus \{0\}$, and the following estimates hold uniformly in k for sufficiently large L:

$$\text{const}_1\, k^{-n} \le \varphi(kL^{-1})/\varphi(L^{-1}) \le \text{const}_2\, k^n \qquad \text{for } 1 \le k \le L, \tag{30}$$

$$\text{const}_3\, k^{-1/2} \le \varphi(kL^{-1})/\varphi(L^{-1}) \le \text{const}_4\, k^{1/2} \qquad \text{for } 0 < k \le 1, \tag{31}$$

where const_i, $i = 1, \dots, 4$, $n > 0$, are some constants.

The estimates (28)–(31) imply

$$\int_{-L}^{L} |\hat{f}(k)|^2 \, |k|^2 \, \frac{\varphi(kL^{-1})}{\varphi(L^{-1})} \, dk \underset{L\to\infty}{\to} \int_{-\infty}^{\infty} |\hat{f}(k)|^2 \, |k|^{\alpha} \, dk.$$

From the other side, the integral over $|k| \geq L$ is $o(1)$ since f is a Schwarz function and m is bounded.

Theorem 3 is proven. □

REMARK 7. We learned very recently that similar results to our Theorem 2 have been independently obtained (in the discrete case) by Tomoyuki Shirai and Yoichiro Takahashi in the preprint [30].

APPENDIX

For the convenience of the reader we give here the proof of a rather standard fact.

LEMMA 3. *Let $\{\eta_L\}$ be a family of random variables such that $c_1(\eta_L) = 0$, $c_2(\eta_L) = 1$ and $c_n(\eta_L)$ converges to zero as $L \to \infty$ for all $n \geq N$, where $N < \infty$. Then $\lim_{L\to\infty} c_n(\eta_L) = 0$ for all $n > 2$ and η_L converges in distribution to $N(0,1)$.*

PROOF. Denote $d_L = \max(|c_j(\eta_L)|^{1/j},\ 1 \leq j \leq N-1)$. It is clear that $d_L \geq 1$. Consider the random variable

$$\tilde{\eta}_L = \eta_L / d_L.$$

Since $c_n(\tilde{\eta}_L) = c_n(\eta_L)/d_L^n$ we have $|c_n(\tilde{\eta}_L)| \leq 1$ for all n and $c_n(\tilde{\eta}_L) \to 0$ for $n \geq N$. Consider $(N-1)$-dimensional vector $(c_1(\tilde{\eta}_L), \dots, c_{N-1}(\tilde{\eta}_L))$. Let $(c_1, c_2, \dots, c_{N-1})$ be a limit point. The Marcinkiewicz theorem (see, e.g., [20]) states that if all but a finite number of cumulants of a random variable are nonzero then the random variable must either have a Gaussian distribution or be a constant. In both cases we have $c_j = 0$ for $j > 2$. Therefore $d_L = (c_2(\eta_L))^{1/2} = 1$ for sufficiently large L and $c_n(\eta_L) \underset{L\to\infty}{\to} 0$ for $n > 2$. Convergence of the cumulants of η_L to the cumulants of $N(0,1)$ is equivalent to the convergence of the moments which in turn implies convergence in distribution. □

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