Final Exam
Due: 8pm, June 9, 2018. Upload everything (both proofs and codes) to CANVAS.

Note: You must follow the UC Davis Academic Integrity. In particular, you are not allowed to discuss the exam questions with any people during the exam period.

**Question 1.** For vector \( z \in \mathbb{R}_+^n \), define its \( k \)-norm \((k < n)\) as

\[
\|z\|_{(k)} = \sum_{i=1}^{k} \bar{z}_i,
\]

where \( \bar{z} \) is obtained by sorting the entries of \( z \) in a non-increasing order. That is, \( \|z\|_{(k)} \) is the sum of the largest \( k \) entries of \( z \).

(i) Compute the proximal mapping of \( \|z\|_{(k)} \). (If an equation is involved, explain how to solve it).

(ii) Compute the conjugate function of \( \|z\|_{(k)} \).

**Question 2.** For a non-differentiable convex function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and a convex set \( \mathcal{X} \), consider the following problem:

\[
\min f(x), \quad \text{s.t.} \quad x \in \mathcal{X}.
\]

The projected subgradient method for solving this problem is

\[
\begin{aligned}
    y^{k+1} &= x^k - \eta_k g^k, \quad \text{where} \quad g^k \in \partial f(x^k) \\
    x^{k+1} &= \text{Proj}_{\mathcal{X}}(y^{k+1}).
\end{aligned}
\]

Suppose \( f \) is Lipschitz continuous, i.e.,

\[
|f(x) - f(y)| \leq L\|x - y\|, \forall x, y \in \text{dom} f.
\]

(i) Analyze the convergence rate of

\[
f\left(\frac{1}{t} \sum_{s=1}^{t} x^s\right) - f(x^*),
\]

where \( x^* \) denotes the optimal solution.

(ii) For fixed step size \( \eta_k = \eta \), what is the best \( \eta \) to choose?
Question 3. Consider the following problem

\[
\begin{align*}
\min_{x} & \quad c^\top x \\
\text{s.t.} & \quad Ax = Az, \\
& \quad \|D^{-1}(x - z)\|_2 \leq \beta,
\end{align*}
\]

where \(A \in \mathbb{R}^{m \times n}\) has full row rank, \(z \in \mathbb{R}^n\) is a given vector with all entries being positive, \(D\) is a diagonal matrix with positive diagonal elements \(D_{ii} := z_i, i = 1, \ldots, n\), and \(\beta \in (0, 1)\).

(i) Give the Karush-Kuhn-Tucker optimality conditions for this problem.

(ii) From the optimality conditions express the optimal solution \(x^*\) as \(x^* = z + p\); i.e., what is \(p\)?

Question 4. Consider the following problem

\[
(P4) \quad \min_{X \in \mathbb{S}_+^{n \times n}} \quad -\log \det X + \|AX - B\|_1,
\]

where \(\mathbb{S}_+^{n \times n}\) denotes the set of \(n \times n\) symmetric positive definite matrices, \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times n}\) are given.

(i) Reformulate this problem so that the reformulation can be solved by linearized ADMM. Write down the reformulation, the updating formula of the linearized ADMM, and the closed-form solution for each of the subproblem. Note: the two subproblems in linearized ADMM must correspond to easily computable proximal mappings.

(ii) Implement your linearized ADMM (write codes). Generate \(A\) and \(B\) randomly for \(n = 10\). Terminate your code when it does not make much progress. Plot the objective value versus the iteration number.

(iii) Now suppose you want to use ADMM instead of linearized ADMM. You need to reformulate (P4) as a 3-block problem, i.e., there are 3 variables, so that you can apply 3-block ADMM to solve the reformulation. Write down the reformulation, the updating formula of the 3-block ADMM, and the closed-form solution for each of the subproblem. Note: the three subproblems must be easy (solving a system of linear equations is considered to be easy in this case).

(iv) Implement your 3-block ADMM (write codes). For the same data \(A\) and \(B\), compare your linearized ADMM and 3-block ADMM by plotting objective values versus iteration number in one figure.